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Matroids, Chow Rings and Log-Concavity

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(Xiangying Chen)

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Chapter 1

Introduction

1.1 Chromatic Polynomials

Let $G = (V, E)$ be a graph. A *proper k-coloring* of G is a map $c : V \to [k]$ such that $c(v) \neq c(w)$ whenever $\{v, w\} \in E$. The number $\chi_G(q)$ of proper q-colorings of G is a polynomial function in q called the *chromatic polynomial* of G. Motivated by the Four Color Problem conjectured in 1852, the chromatic polynomial was introduced by Birkhoff [Bir12] for planar graphs and generally by Whitney [Whi32] and well-studied for over a century. Despite the fact that it did not give a simple proof for 4-CT, the researches on chromatic polynomials gave rise to new fields in mathematics, with many theorems and, certainly, many new conjectures. One of the most notorious conjectures is the unimodality conjecture by Read [Rea68] in 1968, which states that the signless coefficients of the chromatic polynomial of any graph is unimodal. It was conjectured later by Hoggar [Hog74] that these coefficients are log-concave, which is a stronger property than unimodality.

Observe the chromatic polynomial once more. We denote by $G\backslash e$ and G/e the deletion and contraction of a graph $G = (V, E)$ at an edge $e \in E$, respectively. For any proper k-colorings of $G\backslash e$, the two vertices of e are colored either same or differently. These two kinds of colorings correspond to the proper k-colorings of G/e resp. G. Therefore we have the recurrence relation

$$\chi_G(q) = \chi_{G\backslash e}(q) - \chi_{G/e}(q) \tag{1.1}$$

with the initial condition $\chi_{(V,\emptyset)}(q) = q^{|V|}$.

For an edge subset $F \subseteq E$, denote by $G\langle F\rangle = (V, F)$ the subgraph of G

spanned by F, and $k(G\langle F\rangle)$ the number of connected components of $G\langle F\rangle$. Then $q^{k(G\langle F\rangle)}$ counts the q-colorings of G such that the two vertices of $e \in F$ are colored same. By the inclusion-exclusion principle, we have

$$\chi_G(q) = \sum_{F \subseteq E} (-1)^{|F|} q^{k(G\langle F\rangle)}. \tag{1.2}$$

These two formulae ensure that $\chi_G(q)$ is indeed a polynomial function in q, and moreover, they show that the chromatic polynomial and the unimodality and log-concavity conjectures can be generalized to matroids.

1.2 Unimodality and Log-Concavity

A sequence $a_0, \ldots, a_d$ of real numbers is called *unimodal* if there is an $i \in \{0, \ldots, d\}$ such that $a_0 \leq a_1 \leq \ldots \leq a_i \geq \ldots \geq a_d$ and *logarithmically concave* (*log-concave* for short) if $a_i^2 \geq a_{i-1}a_{i+1}$ for any $i \in \{1, \ldots, d-1\}$. It is clear that log-concavity implies unimodality if all elements a_i are positive. We refer to the surveys [Sta89; Bre88; Brä15] for various problems and techniques.

Unimodal and log-concave sequences occur frequently in combinatorics. Binomial coefficients $\binom{n}{k}$, Stirling numbers of the first kind $\left[{n \atop k}\right]$ and the second kind $\left\{{n \atop k}\right\}$ and Eulerian numbers $\left\langle{n \atop k}\right\rangle$ are known to be log-concave hence unimodal for fixed n and $k = 1, \ldots, n$. By a theorem of Newton, if all roots of a polynomial $p(x) = \sum_{i=0}^n a_i x^i$ are real, then $(a_k/\binom{n}{k})_k$ is log-concave, implying that $(a_k)_k$ is log-concave. The Eulerian polynomial, the characteristic polynomial of a real symmetric matrix, the matching polynomial of a graph, and more generally, the independence polynomial of a claw-free graph are proven to be real-rooted, therefore, their coefficients form log-concave sequences.

The unimodality and log-concavity are important for classification, as they give a lot of inequalities. However, these properties behave mysterious, many sequences are proven to be unimodal or log-concave, but many are disproved and many are only conjectured to be so.

The f-vectors of convex polytopes were conjectured to be unimodal in late 1950's, and proven by Björner to be increased on the first quarter and decreased on the last quarter. However, a simplicial polytope of dimension 20 was constructed whose f-vector fails to be unimodal, see [Zie12]. These results are consequences of the g-theorem, whose proof is based on the hard Lefschetz property of the singular cohomology ring of the toric variety associated to the polytope [Sta80]. The g-theorem implies the unimodality of

h-vectors of simplicial polytopes, in particular, the Ehrhart h^*-vector of a Gorenstein lattice polytope with a regular unimodular triangulation [BR07].

Another celebrated example for log-concavity is the Aleksandrov-Fenchel inequality in convex geometry, see [Sch14]. Two combinatorial applications for matroids and posets are given in [Sta81]. It corresponds to the Khovanskii-Teissier inequality for intersection numbers, which is based the classical Hodge index theorem. These are applications of the Hodge-Riemann relations.

The research on the Read-Hoggar conjecture on the unimodality and log-concavity of coefficients of a chromatic polynomial had little progress in a half century until the breakthrough by Huh [Huh12]. Later in [HK12], the log-concavity was proven for the coefficients of characteristic polynomials of representable matroids by using the Khovanskii-Teissier inequality. In [AHK18], a working Hodge theory beyond the case of realizable matroids was developed, and finally the Heron-Rota-Welsh conjecture was proven.

1.3 Combinatorial Hodge Theory

Let $A^\bullet(X) = \bigoplus_{k=0}^d A^k(X)$ be a graded $\mathbb{R}$-algebra associated to some object X of "dimension" d, equipped with a graded bilinear pairing $P : A^\bullet(X) \times A^{d-\bullet}(X) \to \mathbb{R}$ and a graded linear map $L : A^\bullet(X) \to A^{\bullet+1}(X)$. Let K be a convex cone in the space of linear operators on $A^\bullet(X)$. We expect the following properties from the triple $(A^\bullet(X), P, K)$:

(PD) For every $0 \leq k \leq \lfloor \frac{d}{2} \rfloor$, the bilinear pairing $P : A^k(X) \times A^{d-k}(X) \to \mathbb{R}$ is non-degenerate.

(HL) For every $0 \leq k \leq \lfloor \frac{d}{2} \rfloor$ and every $L \in K$, the composition $L^{d-2k} : A^k(X) \to A^{d-k}(X)$ is bijective.

(HR) For every $0 \leq k \leq \lfloor \frac{d}{2} \rfloor$ and every $L \in K$, the bilinear form

$$A^k(X) \times A^k(X) \to \mathbb{R}, \quad (x_1, x_2) \mapsto (-1)^k P(x_1, L^{d-2k} x_2)$$

is symmetric, and is positive definite on the kernel of $L^{d-2k+1} : A^k(X) \to A^{d-k+1}(X)$.

The properties (PD), (HL) and (HR) are called Poincaré duality, hard Lefschetz property and Hodge-Riemann relations, respectively. Together they are called the Kähler package. These properties are enjoyed by the cohomology on a compact Kähler manifold. Huh believed that behind any log-concave sequence that appears in nature, there is such a "Hodge structure" responsible for the log-concavity [Huh16]. The following example is McMullen's polytope algebra [McM89].

Example 1.1 (polytope algebra)**.** Let Π be the abelian group with generators $[P]$, one for each polytope $P \subset \mathbb{R}^d$, which satisfy the following relations:

1. $[P_1 \cup P_2] + [P_1 \cap P_2] = [P_1] + [P_2]$ whenever $P_1 \cup P_2$ is a polytope,
2. $[P + t] = [P]$ for every point $t \in \mathbb{R}^d$, and
3. $[\emptyset] = 0$.

We define a multiplication in Π by the Minkowski sum $[P_1] \cdot [P_2] = [P_1 + P_2]$, and this makes Π a commutative ring with $1 = [\text{point}]$ and $0 = [\emptyset]$, called the *polytope algebra*. One can show that $([P] - 1)^{d+1} = 0$ for any nonempty polytope P. This makes the definition of logarithm on $\Pi \backslash \{0\}$ in the usual way possible, which satisfy the usual rule $\log[P_1 + P_2] = \log[P_1] + \log[P_2]$. As a translating invariant valuation, the Lebesgue measure on $\mathbb{R}^n$ defines a surjective group homomorphism $\mathrm{Vol} : \Pi \to \mathbb{R}$, $[P] \mapsto \mathrm{Vol}(P)$. Moreover, we have the identity

$$\mathrm{Vol}(P) = \frac{1}{d!}\mathrm{Vol}((\log[P])^d)$$

and it can be generalized to Minkowski's mixed volume $\mathrm{Vol}(P_1, \dots, P_d) = \mathrm{Vol}(\log[P_1] \cdots \log[P_d])$ of polytopes $P_1, \dots, P_d$.
We say that the polytopes P_1 and P_2 are *equivalent* if $P_1 \preceq P_2 \preceq P_1$, where $P_1 \preceq P_2$ means that P_1 is a Minkowski summand of some positive multiple of P_2. Let $K(P)$ be the set of all polytopes equivalent to a given polytope P. For each integer $k > 0$, let $\Pi^k(P) \subseteq \Pi$ be the subgroup generated by all elements of the form $p_1 p_2 \cdots p_k$, where p_i is the logarithm of a polytope in $K(P)$. In this sense, $K(P)$ forms a convex cone. In [McM93] it is proven that when P is a d-dimensional simple polytope, the triple $(\Pi^\bullet(P), \mathrm{Vol}, K(P))$ satisfies the desired properties: Let p be the logarithm of a simple polytope in $K(P)$, and let $1 \leq k \leq \frac{d}{2}$.

(PD) The multiplication in Π defines a non-degenerate bilinear pairing $\Pi^k(P) \times \Pi^{d-k}(P) \to \mathbb{R}$, $(x, y) \mapsto \mathrm{Vol}(xy)$.

(HL) The multiplication by p^{d-2k} defines an isomorphism of abelian groups $\Pi^k(P) \to \Pi^{d-k}(P)$, $x \mapsto p^{d-2k}x$.

(HR) The multiplication by p^{d-2k} defines a symmetric bilinear form

$$\Pi^k(P) \times \Pi^k(P) \to \mathbb{R}, \quad (x_1, x_2) \mapsto (-1)^k \mathrm{Vol}(p^{d-2k} x_1 x_2)$$

that is positive definite on the kernel of $p^{d-2k+1} : \Pi^k(P) \to \Pi^{d-k+1}(P)$.

The group $\Pi^k(P)$ can be equipped with a finite dimensional $\mathbb{R}$-vector space structure. The hard Lefschetz theorem (HL) is the main ingredient in the proof of g-conjecture for simple polytopes [Sta80]. The Hodge-Riemann relations (HR) in the spacial case $k = 1$ is essentially equivalent to the Aleksandrov-Fenchel inequality on mixed volumes of convex bodies

$$\mathrm{Vol}(p_1p_1p_3 \cdots p_d)\mathrm{Vol}(p_2p_2p_3 \cdots p_d) \leq \mathrm{Vol}(p_1p_2p_3 \cdots p_d)^2.$$

1.4 Structure of the Thesis

The Heron-Rota-Welsh conjecture was first proven in [AHK18], and the proof in this thesis is based on [BES19], without using order filters. The surveys [Bak18; Huh16; Kat16] provide a nice introduction on these papers. For background in various fields, we refer to [Oxl06; Wel10; Whi86; Whi92; Whi87] for matroid theory, [Zie12] for polytopes, [CLO13] for classical algebraic geometry, [CLS11] for toric varieties and [MS15] for tropical geometry.

In Chapter 2, we present the essential and motivational statements for matroids. In §2.1, the necessary definitions and facts for matroids are given with an emphasis on lattices of flats and matroid polytopes. In §2.2, the expressions of characteristic polynomials of matroids are presented. We state the log-concavity conjecture for Whitney numbers, which are coefficients of characteristic polynomials and are shown to be alternating in sign, and its relation to the log-concavity of the f-vector of the independence complex. In §3.3, a fan structure called the Bergman fan is defined on a matroid. We follow its motivation in tropical geometry that the Bergman fan is a simplicial fan structure on the tropicalization of a realization of a matroid, and it can be generalized to matroids that are not linear and behaves like a linear space.

In Chapter 3, we focus on the Chow ring of a matroid M, which is the Chow cohomology ring of the toric variety associated to the Bergman fan Σ_M of M. In §3.1, we define the Chow ring $A^\bullet(\Sigma)$ and the Minkowski weights $\mathrm{MW}_\bullet(\Sigma)$ of a smooth fan Σ, which are analogue to the cohomology and homology rings in algebraic topology. We give the properties of Chow rings and Minkowski weights, including the Kronecker duality, and the Poincaré duality for complete fans. These properties enable the definition of a cap product and a degree map $\int$. In §3.2, we consider the Chow ring $A^\bullet(M) = A^\bullet(\Sigma_M)$ of a matroid M and study the properties of $A^\bullet(M)$ and its presentations as

$A^\bullet_{\mathrm{FY}}(M)$ and $A^\bullet_\nabla(M)$. A monomial $\mathbb{R}$-basis of $A^\bullet_\nabla(M)$, called the nested basis is given. In §3.3, it is shown that nested basis acquires a combinatorial interpretation as a certain family of matroid quotients called the relative nested quotients. Moreover, the cap product map $A^c_\nabla(M) \xrightarrow{\cdot\cap\Delta_M} \mathrm{MW}_{d-c}(\Sigma_M)$ induces a bijection between the nested basis of $A^\bullet_\nabla(M)$ and the set of relative nested quotients of M, and the bijection respects linear independence.

In Chapter 4, a Hodge theory of matroids sufficient to prove the log-concavity conjecture is developed. The first component of the Kähler package for $A^\bullet(M)$, called the Poincaré duality, is proven in Section 4.1. We show that the Chow ring $A^\bullet(M)$ is the Δ_M-transport of $\mathrm{MW}^\bullet(\Sigma_{A_n}) \cong A^\bullet(\Sigma_{A_n})$, that is, $A^\bullet(M) \cong A^\bullet(\Sigma_{A_n})/\mathrm{Ann}(\Delta_M)$. And the transport structure preserves the Poincaré duality property. As a consequence, the cap product map $A^c(M) \xrightarrow{\cdot\cap\Delta_M} \mathrm{MW}_{d-c}(\Sigma_M)$ is indeed an isomorphism of $\mathbb{R}$-vector spaces.

In Section 4.2, we get a combinatorial expression for the volume polynomial of $A^\bullet_\nabla(M)$. This polynomial is shown to be Lorentzian, which satisfies properties analogue to the Hodge-Riemann relations. In Section 4.3, the Hodge-Riemann relations (HR) and the hard Lefschetz property (HL) for $A^\bullet(M)$ in degree at most 1 are proven. From Section 4.3, we get some element in the ample cone $\mathfrak{K}_M$ for which (HR) is satisfied, it allows the implication of (HR) by (HL). By the facts that tensor products preserve Poincaré duality property and Hodge-Riemann relations, and that the (HR) of transports implies (HL), the required properties follow from induction on the rank of M by $A^\bullet(M)/\mathrm{Ann}(x_F) \cong (A(M|F) \otimes A(M/F))^\bullet$.

At last, in Section 4.4, we show that the Kähler package of $A^\bullet(M)$ of degree at most one implies the log-concavity of Whitney numbers. It is implied by the log-concavity of coefficients $\mu^k(M)$ of the reduced characteristic polynomial $\overline{\chi}_M(q) = \chi_M(q)/(q-1)$. The coefficient $\mu^k(M)$ is exactly the number of initial descending k-step flags of nonempty proper flats of M, and we have $\mu^k(M) = \int_M \alpha^{d-k}\beta^k$. The log-concavity follows then from induction on the rank of M by truncation and the initial case is given by (HL) and (HR) of $(A^\bullet(M), \int_M, \mathfrak{K}_M)$ in degree at most 1.

Chapter 2

Matroids

2.1 Axioms, Operations and Representations

"Anyone who has worked with matroids has come away with the conviction that matroids are one of the richest and most useful mathematical ideas of our day. It is as if one were to condense all trends of present day mathematics onto a single structure, a feat that any would *a priori* deem impossible, were it not for the fact that matroids do exist." So wrote Rota in [Rot08]. Indeed, due to the unique "cryptomorphism" – the very nontrivial equivalence between axiom systems from wholly different mathematical pedigrees, the manipulability and the wide generality and applicability to many well-behaved combinatorial structures, matroids become one of the most elegant structures in mathematics.

Definition 2.1. A *matroid* M on a finite set E is a function $\mathrm{rk}_M : 2^E \to \mathbb{N}$, called the *rank function* of M, which satisfies the following properties:

(R1) If $A \subseteq E$, then $0 \leq \mathrm{rk}_M(A) \leq |E|$.

(R2) If $A \subseteq B \subseteq E$, then $\mathrm{rk}_M(A) \leq \mathrm{rk}_M(B)$.

(R3) If $A, B \subseteq E$, then $\mathrm{rk}_M(A \cup B) + \mathrm{rk}_M(A \cap B) \leq \mathrm{rk}_M(A) + \mathrm{rk}_M(B)$.

We call $\mathrm{rk}(M) := \mathrm{rk}_M(E)$ the *rank* of matroid M. A subset $A \subseteq E$ is called

- *independent* if $\mathrm{rk}_M(A) = |A|$ and *dependent* otherwise,
- *spanning* if $\mathrm{rk}_M(A) = \mathrm{rk}(M)$,
- a *basis* of M if A is independent and spanning,

- a *circuit* if it is a minimal dependent set,
- a *hyperplane* if it is a maximal non-spanning set,
- an *atom* if $\mathrm{rk}_M(A) = 1$,
- a *flat* or a *closed set* if $\mathrm{rk}_M(A \cup \{x\}) = \mathrm{rk}_M(A) + 1$ for any $x \in E \backslash A$.

We denote by $\mathcal{I}(M)$, $\mathcal{S}(M)$, $\mathcal{B}(M)$, $\mathcal{C}(M)$, $\mathcal{H}(M)$ and $\mathfrak{A}(M)$ the sets of independent sets, spanning sets, bases, circuits, hyperplanes and atoms of M, respectively.
For a subset $A \subseteq E$, the *closure* $\mathrm{cl}_M(A)$ of A in M is the minimal flat containing A.

The following are the basic operations on matroids to get new matroids from old ones.

Definition 2.2. Let M be a matroid on E. The *dual* of M is a matroid M^* on E with bases $\mathcal{B}(M^*) = \{E \backslash B : B \in \mathcal{B}(M)\}$.

It follows from definition that $M^{**} = M$. The independent sets and the circuits of M^* are the complements of the spanning sets and hyperplanes of M, respectively, and vice versa. For $A \subseteq E$, $\mathrm{rk}_{M^*}(A) = |A| - \mathrm{rk}(M) + \mathrm{rk}_M(E \backslash A)$.

Definition 2.3. Let M be a matroid on E and $A \subseteq E$. The *restriction* $M|A$ of M to A is a matroid on $E \backslash A$ with independent sets $\mathcal{I}(M|A) = \{I \in \mathcal{I}(M) : I \subseteq A\}$. The *deletion* of A from M is the matroid $M \backslash A := M|(E \backslash A)$. The *contraction* M/A of A from M is the matroid given by $M/A := (M^* \backslash A)^*$. A matroid M' is called a *minor* of M if it can be obtained from M by a sequence of restriction and contraction operations. (Remark that the order is irrelevant.)

Definition 2.4. Let M_1 be a matroid on E_1 and M_2 be a matroid on E_2 where $E_1 \cap E_2 = \emptyset$. The *direct sum* of M_1 and M_2 is the matroid $M_1 \oplus M_2$ on $E_1 \cup E_2$ whose independent sets are $\mathcal{I}(M_1 \oplus M_2) = \{I \cup J : I \in \mathcal{I}(M_1), J \in \mathcal{I}(M_2)\}$.

We state in addition the flat axioms of matroids here because its great relevance in this thesis.

Definition 2.5. A *matroid* M on a finite set E is a collection $\mathcal{L}_M$ of subsets of E, called *flats* of M, satisfying the following properties:

(F1) $E \in \mathcal{L}_M$.

(F2) If $F_1, F_2 \in \mathcal{L}_M$, then $F_1 \cap F_2 \in \mathcal{L}_M$.

(F3) If $F \in \mathcal{L}_M$ and $\{F_1, \ldots, F_k\}$ the set of minimal members of $\mathcal{L}_M$ that properly contain F, then the sets $F_1 \backslash F, \ldots, F_k \backslash F$ partition $E \backslash F$.

The set of flats $\mathcal{L}_M$ of a matroid M, ordered by inclusion, is a lattice with meet and join

$$A \wedge B = A \cap B, \quad A \vee B = \mathrm{cl}_M(A \cup B),$$

called the *lattice of flats* of M.

Definition 2.6. Let M be a matroid on E. An element $e \in E$ is called a *loop* in M if $\mathrm{rk}_M(\{e\}) = 0$, and it is called a *coloop* in M if it is a loop in M^*, i.e. if $\mathrm{rk}_M(E \backslash e) = \mathrm{rk}(M) - 1$. A matroid M is called *simple* or a *combinatorial geometry* if it is loopless and every atom of M has cardinality 1.

One can see that for every matroid M, there is a simple matroid on $\mathfrak{A}(M)$ with lattice of flats isomorphic to $\mathcal{L}_M$: If we delete all loops from M and then, for each atom $A \in \mathfrak{A}(M)$ with $|A| \geq 2$, delete all but one distinguished element of A, the matroid we obtain is unique up to a renaming the distinguished elements. The following is a classification for lattices of flats of matroids.

Definition 2.7. A lattice is *atomic* if every element is a join of atoms. A graded lattice is *semimodular* if its rank function r satisfies

$$r(x) + r(y) \geq r(x \vee y) + r(x \wedge y) \ \forall x, y.$$

A *geometric lattice* is a finite atomic semimodular lattice.

Theorem 2.1. *A lattice is geometric iff it is the lattice of flats of a matroid.*

That is, geometric lattices are cryptomorphic to simple matroids: Given a geometric lattice $\mathcal{L}$, we can define a simple matroid M on the set of atoms of $\mathcal{L}$ with the rank function that maps A to the rank of $\bigvee A$ in $\mathcal{L}$, i.e. the length of any chain from the minimum to $\bigvee A$ in $\mathcal{L}$. Then we have $\mathcal{L}_M \cong \mathcal{L}$. Remark that every geometric lattice is also coatomic. By definition, the coatoms of a matroid M are the hyperplanes of M.

Proposition 2.2. *Let F be a flat in a matroid M and suppose that* $\mathrm{rk}_M(F) = \mathrm{rk}(M) - k$. *Then M has a set $\{H_1, \ldots, H_k\}$ of hyperplanes such that $F = \bigcap_{i=1}^k H_i$.*

Therefore, by duality, every flat of the dual matroid M^* is the complement of a union of circuits in M.

The following statements are the interaction of minor operations on the lattice of flats, which is easy to show but relevant to the induction of the main theorem in this thesis.

Proposition 2.3. *Let M be a matroid on E and $A \subseteq M$. Let $F \subseteq E \backslash A$ be a subset. Then $F \in \mathcal{L}_{M/A}$ iff $F \cup A \in \mathcal{L}_M$, and $F \in \mathcal{L}_{M\backslash A}$ iff $F = F' \backslash A$ for some flat F' of M.*

Corollary 2.4. *Let M be a matroid on E and $A \subseteq E$. Then*

1. *$\mathcal{L}_{M/A}$ is isomorphic to the interval $[\mathrm{cl}_M(A), E]$ of $\mathcal{L}_M$,*
2. *if A is a flat of M, then $\mathcal{L}_{M|A}$ is isomorphic to the interval $[\mathrm{cl}_M(\emptyset), A]$ of $\mathcal{L}_M$.*

Another point of view on matroids is by polytopes. The optimization problem on matroids can be reformulated by a linear programming problem on matroid polytopes. Remark that matroids can also be characterized by the property that the greedy algorithm always finds an optimal solution.

Definition 2.8. Let M be a matroid on E. The *(matroid) base polytope* $\mathcal{P}(M)$ of M is a polytope in $\mathbb{R}^E$ defined by

$$\mathcal{P}(M) := \mathrm{conv}\,\{\mathbf{e}_B : B \in \mathcal{B}(M)\} \subset \mathbb{R}^E,$$

where $\mathbf{e}_B := \sum_{i \in B} \mathbf{e}_i$.

Proposition 2.5. *The hyperplane-representation of the base polytope $\mathcal{P}(M)$ of a matroid M on E is*

$$\mathcal{P}(M) = \left\{ \mathbf{x} \in \mathbb{R}^E \;\middle|\; x_i \geq 0 \;\forall i \in E,\; \sum_{i \in S} x_i \leq \mathrm{rk}_M(S) \;\forall S \subseteq E,\; \sum_{i \in E} x_i = \mathrm{rk}(M) \right\}.$$

Proposition 2.6. 1. *$\mathcal{P}(M^*) = \mathbf{1} - \mathcal{P}(M)$ where $\mathbf{1} = (1, \ldots, 1)$.*

2. *$\mathcal{P}(M_1 \oplus M_2) = \mathcal{P}(M_1) \times \mathcal{P}(M_2)$.*
3. *Let M be a matroid on E and $e \in E$. If e is not a loop or a coloop, then $\mathcal{P}(M \backslash e) \cong \mathcal{P}(M) \cap \{\mathbf{x} : x_e = 0\}$ and $\mathcal{P}(M/e) \cong \mathcal{P}(M) \cap \{\mathbf{x} : x_e = 1\}$. If e is a loop or a coloop, then $\mathcal{P}(M \backslash e) = \mathcal{P}(M/e) \cong \mathcal{P}(M)$, as $\mathcal{P}(M)$ lies in the hyperplane $\{\mathbf{x} : x_e = 0\}$ or $\{\mathbf{x} : x_e = 1\}$, respectively.*

The following characterization of matroid polytopes can be regarded as another cryptomorphic definition of matroids.

Theorem 2.7 ([Gel+87]). *A polytope $P \subset \mathbb{R}^E$ is the base polytope of a matroid on E iff*

(P1) every vertex of P is in $\{0,1\}^E$,

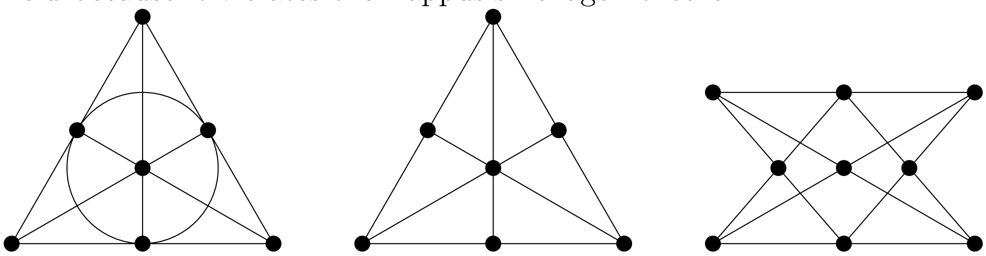
(P2) every edge of P is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i, j \in E$.

Example 2.8 (uniform matroids). Let m, n be non-negative integers such that $m \leq n$. Let E be an n-element set. The set $\binom{E}{m}$ of m-element subsets of E is the set of bases of a matroid on E, called the *uniform matroid* of rank m on E, denoted by $U_{m,E}$ or $U_{m,n}$. For $A \subseteq E$, we have $\mathrm{rk}_{U_{m,n}}(A) = \min\{|A|, m\}$. The set of flats of $U_{m,n}$ consist of all subsets $A \subseteq E$ of cardinality less than m and E itself. The dual of $U_{m,n}$ is $U_{n-m,n}$. The base polytope $\mathcal{P}(U_{m,n})$ is the hypersimplex $\Delta_{n,m}$. The matroid $U_{n,n}$ is called the *Boolean matroid* because its lattice of flats is a Boolean lattice.

Example 2.9 (graphic matroids). Let $G = (V, E)$ be a graph. The set of cycles of G is the set of circuits of a matroid on E, called the *cycle matroid* $M(G)$ of G. The bases of $M(G)$ are the spanning forests of G, the hyperplanes of $M(G)$ are the complements of the minimal cuts in G. The rank of $A \subseteq E$ in $M(G)$ is $|V| - k(G\langle A\rangle)$. The flats of $M(G)$ are the edge sets of the induced subgraphs of G. The loops of $M(G)$ are the loops of G, and $M(G)$ is simple iff G is a simple graph. Moreover, the minor operations and the dual are compatible to graph operations by $M(G\backslash e) = M(G)\backslash e$ and $M(G/e) = M(G)/e$, and $M(G)^* = M(G^*)$ for planar graphs G where G^* is a plane dual of G. Remark that non-isomorphic graphs can have the same cycle matroid. A matroid is called *graphic* if it is the cycle matroid of some graph.

Example 2.10 (linear matroids). Let A be an $m \times n$-matrix over the field $\mathbb{K}$. Let E be the set of column labels of A. Then $M(A)$ is a matroid on E, called the *column matroid* of A, whose independent sets are the subsets of E indexing linearly independent multisets of column vectors. A matroid is called *linear* or *representable* over $\mathbb{K}$ if it is the column matroid of some matrix over $\mathbb{K}$, and is called *regular* if it is linear over any field. Every graphic matroid $M = M(G)$ is regular because it is the column matroid of the incidence matrix of an arbitrary orientation of G. A matrix A can be regarded as a vector configuration or a hyperplane configuration consisting of the column vectors resp. their orthogonal complements. Deletion and contraction by e correspond to removing e and projecting to the hyperplane with normal vector e, respectively. Duality is the orthogonal complementarity of the row spaces, explicitly, $M([I_r|D])^* = M([-D^\top|I_{n-r}])$.

Example 2.11 (non-representable matroids)**.** It was proven recently that almost all matroids are not representable over any field [Nel16]. The following pictures are the rank 3 simple matroids M_1, M_2 and M_3, where points and lines mean rank 1 and 2 flats, respectively. The Fano matroid $M_1 = \mathbb{P}\mathbb{F}_2^2$ is representable over $\mathbb{K}$ iff $\mathrm{char}(\mathbb{K}) = 2$, and the non-Fano matroid M_2 is representable over $\mathbb{K}$ iff $\mathrm{char}(\mathbb{K}) \neq 2$. Therefore $M_1 \oplus M_2$ is not representable over any field. The non-Pappus matroid M_3 is not representable over any field because it violates the Pappus's hexagon theorem.



M_1 M_2 M_3

2.2 Characteristic Polynomials

As we have seen in Example 2.9, matroids are a generalization of graphs. Actually, almost all graph-theoretic statements without concerning vertices can be rephrased in matroid theoretic language. As vertices do not play an essential role in in the representations (1.1) and (1.2) of chromatic polynomials, we can generalize them to matroids as follows.

Definition 2.9. Let M be a matroid on E. The *characteristic polynomial* (or *chromatic polynomial*) of M is defined by

$$\chi_M(q) = \sum_{A \subseteq E} (-1)^{|A|} q^{\mathrm{rk}(M) - \mathrm{rk}_M(A)}.$$

If the matroid $M = M(G)$ is graphic, then from $\mathrm{rk}_M(A) = |V| - k(G\langle A \rangle)$ and $\mathrm{rk}(M) = |V| - k(G)$ for any $A \subseteq E$, we have

$$\chi_G(q) = q^{k(G)} \chi_M(q).$$

It is not surprising that the characteristic polynomial also satisfies the recurrence relation similar to (1.1).

Proposition 2.12. *Let $\chi_M(q)$ be the characteristic polynomial of a matroid M on E and $e \in E$ not a coloop. Then*

$$\chi_M(q) = \chi_{M\backslash e}(q) - \chi_{M/e}(q).$$

If $e \in E$ is a coloop, then

$$\chi_M(q) = (q-1)\chi_{M\backslash e}(q).$$

If $M = M_1 \oplus M_2$, then

$$\chi_M(q) = \chi_{M_1}(q)\chi_{M_2}(q).$$

Remark that $M/e = M\backslash e$ for a loop or a coloop $e \in E$, thus $\chi_M(q) = 0$ for a matroid M containing a loop. That reflects the fact that a graph with loops has no proper coloring.

Let M be a matroid on E. Let $\mathcal{L}_M$ be the lattice of flats of M with Möbius function $\mu_M : \mathcal{L}_M \times \mathcal{L}_M \to \mathbb{Z}$. Recall that the Möbius function of a poset is given recursively by

$$\mu(x,y) = \begin{cases} 0 & x \not\leq y \\ 1 & x = y \\ -\sum_{x \leq z < y} \mu(x,z) & x < y \end{cases}.$$

The Möbius function μ_M can be expanded by the Möbius function of the Boolean lattice on E as

$$\mu_M(F_1, F_2) = \sum_{\substack{F_1 \subseteq A \subseteq F_2 \\ \mathrm{cl}_M(A) = F_2}} (-1)^{|A|-|F_1|}.$$

One can verify easily that this expansion satisfies the definition of μ_M. Therefore, we can collect the terms of the characteristic polynomial as follows.

Proposition 2.13. *Let M be a loopless matroid. Then we have*

$$\chi_M(q) = \sum_{F \in \mathcal{L}_M} \mu_M(\emptyset, F) q^{\mathrm{rk}(M)-\mathrm{rk}(F)}. \tag{2.1}$$

The coefficient $w_k(M)$ of $q^{\mathrm{rk}(M)-k}$ in $\chi_M(q)$ is called the *k-th Whitney number of the first kind* of M, that is,

$$\chi_M(q) = \sum_{k=0}^{\mathrm{rk}(M)} w_k(M) q^{\mathrm{rk}(M)-k}, \qquad w_k(M) = \sum_{\substack{F \in \mathcal{L}_M \\ \mathrm{rk}_M(F) = k}} \mu_M(\emptyset, F).$$

The following theorem implies that $(-1)^k w_k(M)$ is positive for any loopless matroid M, which makes the conjecture for log-concavity meaningful.

Theorem 2.14 ([Rot64])**.** *The Möbius function $\mu_{\mathcal{L}}$ of a geometric lattice $\mathcal{L}$ is nonzero and alternates in sign. Precisely,*

$$(-1)^{r(y)-r(x)}\mu_{\mathcal{L}}(x,y) > 0$$

if $x \leq y$ in $\mathcal{L}$, where r is the rank function in $\mathcal{L}$.

Proof. Since every interval of a geometric lattice is geometric it suffices to prove that $(-1)^{\mathrm{rk}(M)}\mu_M(\emptyset, E) > 0$ for a simple matroid M on E. We prove by induction on the rank $\mathrm{rk}(M)$ and the corank $|E| - \mathrm{rk}(M)$ of M.
If M has corank 0, it is a Boolean matroid and $\mathcal{L}_M$ is a Boolean lattice. Hence $\mu_M(\emptyset, E) = (-1)^{|E|}$ and $(-1)^{\mathrm{rk}(M)}\mu_M(\emptyset, E) = 1 > 0$. This case includes simple matroids with rank 0 or 1.
If M has positive corank, it is not a Boolean matroid. Hence there is an element $e \in E$ which is not a coloop. By induction on rank, $(-1)^{\mathrm{rk}(M/e)}\mu_{M/e}(\emptyset, E\backslash e) > 0$. By induction on corank, $(-1)^{\mathrm{rk}(M\backslash e)}\mu_{M\backslash e}(\emptyset, E\backslash e) > 0$. By Proposition 2.12, the constant terms in the expression (2.1) satisfy

$$(-1)^{\mathrm{rk}(M)}\mu_M(\emptyset, E) = (-1)^{\mathrm{rk}(M/e)}\mu_{M/e}(\emptyset, E\backslash e) + (-1)^{\mathrm{rk}(M\backslash e)}\mu_{M\backslash e}(\emptyset, E\backslash e) > 0,$$

which completes the proof. □

Corollary 2.15. *The Whitney numbers $w_k(M)$ of a loopless matroid M are nonzero and alternate in sign. Precisely,*

$$(-1)^k w_k(M) = |w_k(M)| > 0.$$

Conjecture 2.16 (Heron-Rota-Welsh conjecture)**.** *The sequence $w_k(M)$ is log-concave:*

$$w_{k-1}(M)w_{k+1}(M) \leq w_k(M)^2 \quad \textit{for all } 1 \leq k \leq \mathrm{rk}(M) - 1.$$

In particular, $w_k(M)$ is unimodal:

$$|w_0(M)| \leq \cdots \leq |w_l(M)| \geq \cdots \geq |w_{\mathrm{rk}(M)}(M)| \quad \textit{for some } 0 \leq l \leq \mathrm{rk}(M).$$

Because the characteristic polynomial is the protagonist in the main theorem of the thesis, we state here some important properties and applications of the characteristic polynomials.

The *beta invariant* $\beta(M)$ of a matroid M is given by

$$\beta(M) = (-1)^{\mathrm{rk}(M)-1} \frac{\mathrm{d}}{\mathrm{d}q} \chi_M(q) \bigg|_{q=1} = (-1)^{\mathrm{rk}(M)} \sum_{F \in \mathcal{L}_M} \mu_M(\emptyset, F) \mathrm{rk}_M(F)$$

This numerical invariant of matroids deduced from characteristic polynomial encodes much nontrivial structural information of matroid.

Theorem 2.17 ([Cra67; Bry71]). *Let M be a matroid.*

- $\beta(M) \geq 0$ *and* $\beta(M) > 0$ *iff M is connected and is not a loop.*
- $\beta(M) = \beta(M \backslash e) + \beta(M/e)$ *if e is neither a loop nor a coloop, and* $\beta(M^*) = \beta(M)$ *if M is not a loop or a coloop.*
- *A matroid M is regular iff $\beta(M_1) \leq 1$ for all four element minors M_1 of M and $\beta(M_2) \leq 2$ for all seven element minors M_2 of M.*
- *A matroid M is the cycle matroid of a series-parallel network (graphs without K_4 as minor) iff it is not a coloop and $\beta(M) = 1$.*

The characteristic polynomial is a *Tutte-Grothendieck invariant* of matroids, which is an invariant f of matroids satisfying $f(M) = f(M \backslash e) + f(M/e)$ for e neither a loop nor a coloop, and $f(M_1 \oplus M_2) = f(M_1)f(M_2)$. The *Tutte polynomial* $T_M(x, y)$ of a matroid M on E is defined by

$$T_M(x, y) = \sum_{A \subseteq E} (x-1)^{\mathrm{rk}(M) - \mathrm{rk}_M(A)} (y-1)^{|A| - \mathrm{rk}_M(A)}.$$

which has the recurrence relation

$$T_M(x, y) = \begin{cases} T_{M \backslash e}(x, y) + T_{M/e}(x, y) & \text{if } e \text{ is neither a loop nor a coloop,} \\ x T_{M \backslash e}(x, y) & \text{if } e \text{ is a coloop,} \\ y T_{M \backslash e}(x, y) & \text{if } e \text{ is a loop,} \end{cases}$$

It is a universal T-G invariant of matroids, in the sense that every T-G invariant $f(M)$ is an evaluation of the Tutte polynomial $T_M(x, y)$ by setting $x = f(\text{coloop})$ and $y = f(\text{loop})$. By the recurrence relation in Proposition 2.12, the invariant $(-1)^{\mathrm{rk}(M)} \chi_M(q)$ is a T-G invariant, and we have

$$\chi_M(q) = (-1)^{\mathrm{rk}(M)} T_M(1-q, 0)$$

and the beta invariant $\beta(M)$ is the coefficient of $x^0 y^1$ (and $x^1 y^0$) in $T_M(x, y)$. The Tutte polynomial has nice properties such as $T_M(x, y) = T_{M^*}(y, x)$ and the convolution formula

$$T_M(x, y) = \sum_{A \subseteq E} T_{M/A}(x, 0) T_{M|A}(0, y).$$

The independent sets $\mathcal{I}(M)$ of a matroid M on E form a pure simplicial complex on E, called the *independence complex* of M. In particular, a simplicial complex $\mathcal{I}$ on E is the set of independent sets of a matroid iff it satisfies the independence augmentation axiom

(I3) if $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there is an element $e \in I_2 \backslash I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

Another classification for independence complexes is

(I3') $\mathcal{I}|_S := \{I \in \mathcal{I} : I \subseteq S\}$ is pure for all $S \subseteq E$.

Let $(f_0(M), \ldots, f_{\mathrm{rk}(M)}(M))$ be the f-vector of the independence complex $\mathcal{I}(M)$ of a matroid M on E, i.e. $f_k(M) = |\{I \in \mathcal{I}(M) : |I| = k\}|$, and

$$f_M(x) := \sum_{k=0}^{\mathrm{rk}(M)} f_i(M) x^{\mathrm{rk}(M)-k}$$

be its f-polynomial. By comparing with the explicit or recurrence definition of the Tutte polynomial, we have $f_M(x) = T_M(x+1, 1)$.
In [Mas72], Mason proposed the following three conjectures about log-concavity, written in increasing strength.

Conjecture 2.18 (Mason's conjecture)**.** *For any matroid M and any $1 \leq k \leq \mathrm{rk}(M) - 1$,*

(i) $f_k(M)^2 \geq f_{k-1}(M) f_{k+1}(M)$,

(ii) $f_k(M)^2 \geq \left(1 + \frac{1}{k}\right) f_{k-1}(M) f_{k+1}(M)$,

(iii) $f_k(M)^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{f_1(M) - k}\right) f_{k-1}(M) f_{k+1}(M)$

The (i) part of Mason's conjecture follows from the log-concavity of $w_k(M)$ due to the following rediscovery of [Bry77] in [Len13]. The *broken circuits* of a loopless matroid M on a linearly ordered set E are $\{C \backslash \min C : C \in \mathcal{C}(M)\}$. The *broken circuit complex* $\mathrm{BC}(M)$ is the collection of all subsets of E that do not contain a broken circuit. A classical result in [Rot64] states that the f-polynomial of $\mathrm{BC}(M)$ is exactly the signless characteristic polynomial of M. That is,

$$f_{\mathrm{BC}(M)}(q) = (-1)^r \chi_M(-q) \quad \text{and} \quad f_k(\mathrm{BC}(M)) = (-1)^k w_k(M).$$

Let $e \notin E$. The *free coextension* of M is the matroid $(M^* \oplus U_{1,\{e\}})^*$ on $E \cup \{e\}$. Remark that e is a loop in the free coextension and hence is in every set of its broken circuits complex. In [Bry77] it is shown that the independence complex of any matroid M is the broken circuit complex of the free coextension of M restricted in E, that is,

$$\mathcal{I}(M) = \mathrm{BC}((M^* \oplus U_{1,\{e\}})^*)|_E.$$

Therefore we get the following theorem which shows that the log-concavity of $w_k(M)$ implies the log-concavity of $f_k(M)$.

Theorem 2.19. *Let M be a matroid of rank r and let $M' := (M^* \oplus U_{1,1})^*$ be the free coextension of M. Then*

$$(-1)^{r+1}\chi_{M'}(-q) = (1+q)f_M(q).$$

Recently, the strongest form (iii) of Mason's Conjecture is proven independently in [Ana+18] and [BH18].

Example 2.20 (uniform matroids)**.** The lattice of flats of the uniform matroid $U_{m,n}$ on $[n]$ is $\mathcal{L}_{U_{m,n}} = \{F \subseteq [n] : |F| \leq m-1 \text{ or } |F| = n\}$. Therefore, we have

$$\chi_{U_{m,n}}(q) = \sum_{k=0}^{m-1}(-1)^k \binom{n}{k} \left(q^{m-k} - 1\right).$$

The Whitney numbers $w_k(U_{m,n}) = (-1)^k\binom{n}{k}$ are signed binomial coefficients for $k < m$, and when $k = m$, $w_m(U_{m,n}) = -\sum_{k=0}^{m-1}(-1)^k\binom{n}{k} = (-1)^m\binom{n-1}{m-1}$ is a partial sum of them. The log-concavity of $w_k(U_{m,n})$, $k = 1, \ldots, m$ follows from the log-concavity of binomial coefficients and the fact that $\binom{n}{m} \geq \binom{n-1}{m-1}$.

Example 2.21 (linear matroids)**.** Let M be the matroid of a hyperplane arrangement $\mathcal{A}$ in $\mathbb{K}^n$, and $\mathbb{K}^n \backslash \mathcal{A} = \mathbb{K}^n \backslash \bigcup \mathcal{A}$ be its complement.

1. [Zas75] If $\mathbb{K} = \mathbb{R}$, then the number of regions in $\mathbb{R}^n \backslash \mathcal{A}$ is equal to $(-1)^n\chi_M(-1)$, and the number of relatively bounded regions is equal to $(-1)^{\mathrm{rk}(M)}\chi_M(1)$.
2. [OS80] If $\mathbb{K} = \mathbb{C}$, then the Poincaré polynomial (the generating function of Betti numbers) of the cohomology ring of $\mathbb{C}^n \backslash \mathcal{A}$ is given by

$$\sum_{k \geq 0} \operatorname{rank} H^k(\mathbb{C}^n \backslash \mathcal{A}, \mathbb{Z})\, q^k = (-q)^n \chi_M\left(-\frac{1}{q}\right).$$

3. [CR70] If $\mathbb{K} = \mathbb{Q}$, by multiplying each hyperplane equation by a suitable integer and modulo a prime p, we get an induced hyperplane arrangement $\mathcal{A}_q$ in $\mathbb{F}_q^n$, $q = p^r$. For large enough p, the intersection lattices of

$\mathcal{A}$ and $\mathcal{A}_q$ are isomorphic, in this case,

$$\left|\mathbb{F}_q^n \backslash \mathcal{A}_q\right| = q^n - \left|\bigcup \mathcal{A}_q\right| = q^{n-\mathrm{rk}(M)}\chi_M(q).$$

2.3 Bergman Fans of Matroids

Let $\mathbb{K}$ be any field with a valuation. In this section we mostly work with the *trivial valuation*, that is $\mathrm{val}(a) = 0$ for all $a \in \mathbb{K}^*$ and $\mathrm{val}(0) = \infty$. Given a Laurent polynomial $f = \sum_{\mathbf{u}\in\mathbb{Z}^n} c_{\mathbf{u}} x^{\mathbf{u}} \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the *tropicalization* of f is the piecewise linear function $\mathrm{trop}(f) : \mathbb{R}^n \to \mathbb{R}$ given by

$$\mathrm{trop}(f)(\mathbf{w}) = \min_{\mathbf{u}\in\mathbb{Z}^n}\{\mathrm{val}(c_{\mathbf{u}}) + \mathbf{w}\cdot\mathbf{u}\}.$$

$\mathrm{trop}(f)$ is a polynomial in the *tropical semiring* $(\mathbb{R}\cup\{\infty\}, \min, +)$, called a *tropical polynomial.*
Analogue to the classical variety which is the solution set of a polynomial equation system, when F is a tropical polynomial, the set

$$V(F) = \{\mathbf{w} \in \mathbb{R}^n : \text{the minimum in } F(\mathbf{w}) \text{ is achieved at least twice}\}$$

is the locus in $\mathbb{R}^n$ where the piecewise linear function F failed to be linear, called a *tropical hypersurface*. We denote that $\mathrm{trop}(V(f)) = V(\mathrm{trop}(f))$.
Let I be an ideal in $\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and let $X = V(I)$ be its variety in the torus $T^n = (\mathbb{K}^*)^n$. The *tropicalization* of the variety X is

$$\mathrm{trop}(X) = \bigcap_{f\in I} \mathrm{trop}(V(f)) \subseteq \mathbb{R}^n.$$

Note that $\mathrm{trop}(X)$ is generally not the intersection over the tropical hypersurfaces $\mathrm{trop}(V(f))$ where f runs over a generating set of the ideal I of X. We call a finite generating set $\mathcal{T}$ for an ideal I in $\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ a *tropical basis* if

$$\mathrm{trop}(V(I)) = \bigcap_{f\in\mathcal{T}} \mathrm{trop}(V(f)).$$

It is known that every ideal has a finite tropical basis.
In 1971, Bergman [Ber71] introduced the logarithmic limit set of a subvariety of the complex algebraic torus $(\mathbb{C}^*)^n$ which is the same as the tropical variety. Let I be an ideal in $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $V(I) = \{\mathbf{z} \in (\mathbb{C}^*)^n : f(\mathbf{z}) = 0 \text{ for all } f \in I\} \subseteq (\mathbb{C}^*)^n$ be its variety. The *amoeba* of the ideal I is the subset of $\mathbb{R}^n$ defined as image of the coordinate-wise logarithm map:

$$\mathcal{A}(I) = \{(\log(|z_1|), \ldots, \log(|z_n|)) \in \mathbb{R}^n : \mathbf{z} = (z_1, \ldots, z_n) \in V(I)\}.$$

For any real number $M > 0$, consider the set

$$\mathcal{A}_M(I) = -\frac{1}{M}\mathcal{A}(I) \cap S^{n-1}$$

where $S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x}|| = 1\}$ is the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$. The *logarithmic limit set* $\mathcal{A}_\infty(I)$ is the set of points $\mathbf{v}$ on the sphere S^{n-1} such that there exists a sequence of points $\mathbf{v}_M \in \mathcal{A}_M(I)$ converging to $\mathbf{v}$. In other words, it is the asymptotic directions of the tentacles of the amoeba.

Theorem 2.22 ([Jon16])**.** *The tropical variety of I coincides with the cone over the logarithmic limit set $\mathcal{A}_\infty(I)$, i.e., a nonzero vector $\mathbf{w} \in \mathbb{R}^n$ lies in* $\operatorname{trop}(V(I))$ *iff the corresponding unit vector $\frac{1}{||\mathbf{w}||}\mathbf{w}$ lies in $\mathcal{A}_\infty(I)$.*

By the structure theorem of tropical varieties, $\operatorname{trop}(V(I))$ is the support of a polyhedral fan. The fan structure on $\operatorname{trop}(V(I))$ is called the *Bergman fan* of X and it implies that $\mathcal{A}_\infty(I)$ is a spherical polyhedral complex in S^{n-1}, called the *Bergman complex*.

Let $\mathcal{A} = \{H_i : 0 \leq i \leq n\}$ be an arrangement of $n+1$ hyperplanes in the projective space $\mathbb{P}^d$. Namely, $\mathcal{A}$ represents a simple linear matroid M. Let $X = \mathbb{P}^d \setminus \bigcup \mathcal{A}$ be the complement of hyperplane arrangement. We will see that X is naturally a closed subvariety of the torus T^n cut out by a linear system of equations. The tropicalization of X depends only on the combinatorics of $\mathcal{A}$, namely, the matroid M. As X is a linear subspace in T^n, $\operatorname{trop}(X)$ is referred to as *tropicalized linear space*. Moreover, we can get different descriptions for $\operatorname{trop}(X)$, some of them do not depend on the representation of M, thus can be generalized to any matroid, to define the *tropical linear spaces* and their Bergman fans.

Write $H_i = \{\mathbf{z} \in \mathbb{P}^n : \mathbf{b}_i \cdot \mathbf{z} = 0\}$ where $\mathbf{b}_i \in \mathbb{K}^{d+1}$ is a normal vector of the hyperplane H_i. We can assume that $\mathbf{b}_0, \ldots, \mathbf{b}_n$ span $\mathbb{K}^{d+1}$.
Fix a torus $T^n = (\mathbb{K}^*)^{n+1}/\mathbb{K}^*$ in $\mathbb{P}^n$. Consider the map $\iota : X \to T^n$, $\mathbf{z} \mapsto (\mathbf{b}_0 \cdot \mathbf{z}, \ldots, \mathbf{b}_n \cdot \mathbf{z})$. The map ι is injective as $\mathbf{b}_i$ span $\mathbb{K}^{n+1}$. Thus the image $\iota(X)$ is a closed subset of T^n. Let $B = (\mathbf{b}_0 \ \cdots \ \mathbf{b}_n)$ be the $(d+1) \times (n+1)$-matrix whose columns are $\mathbf{b}_i$, and let $A = (a_{ij})$ be an $(n-d) \times (n+1)$-matrix whose rows are a basis for the kernel of B. Namely, the columns of A is a *Gale transform* of $\{\mathbf{b}_0, \ldots, \mathbf{b}_n\}$. Let

$$I = \left\langle f_i = \sum_{j=0}^{n} a_{ij} x_j \,\middle|\, 1 \leq i \leq n-d \right\rangle$$

be the homogeneous ideal in $\mathbb{K}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$. That means $\iota(X) = B^\top X$ and $V(I) = \ker A \cap T^n$. Because of our construction we have $\ker A = \operatorname{im} B^\top$ and

$\iota(X) \subseteq T^n$. If $\mathbf{x} \in V(I) = \ker A \cap T^n$ then each coordinate of $\mathbf{x}$ is nonzero, so the unique point $\mathbf{z} \in \mathbb{K}^{d+1}$ with $\mathbf{x} = B^\top \mathbf{z}$ does not lie in any hyperplane of $\mathcal{A}$, so $\mathbf{x} \in B^\top X$. We have proven that $\iota(X) = V(I)$.

Proposition 2.23. *The map ι defines an isomorphism between the hyperplane arrangement complement $X = \mathbb{P}^d \backslash \bigcup \mathcal{A}$ and the subvariety $V(I)$ of T^n.*

By reversing the construction, we see that any ideal I generated by linear forms arises from some hyperplane arrangement. If the linear forms are not homogeneous, we can homogenize the ideal. So we get a correspondence between hyperplane arrangements $\mathcal{A}$ in $\mathbb{P}^d$ and homogeneous linear ideals $I \subset \mathbb{K}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$: The coefficients of linear forms in I are exactly the dependencies of B.

So where are the information of matroids? In the hyperplane arrangement $\mathcal{A}$ a circuit $C \subseteq \mathcal{A}$ consists of hyperplanes whose normal vectors are minimally dependent. That means for any $j \in C$, $\bigcap_{i \in C} H_i = \bigcap_{i \in C \setminus \{j\}} H_i$ and has codimension $|C| - 1$. As the dependencies are the coefficients of linear forms in I, a circuit C is also the inclusion-minimal support $C = \operatorname{supp}(l_C) = \{i : a_i \neq 0\}$ of a nonzero linear form $0 \neq l_C = \sum a_i x_i \in I$. The linear form l_C is uniquely determined for each circuit C up to scaling.

It can be shown via the Gröbner theory that the circuits form a tropical basis for I.

Proposition 2.24. *Let $I \subseteq \mathbb{K}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$ be an ideal generated by linear forms where $\mathbb{K}$ has the trivial valuation, and consider the hyperplane arrangement $X = V(I)$. The set of polynomials l_C in I whose supports are circuits is a tropical basis for I. Equivalently,*

$$\operatorname{trop}(X) = \left\{ \mathbf{w} \in \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1} \;\middle|\; \min_{i \in C} w_i \textit{ is attained at least twice } \forall \textit{ circuit } C \textit{ of } I \right\}.$$

Note that if $\mathbf{w} \in \operatorname{trop}(X)$ then $\mathbf{w} + \lambda \mathbf{1} \in \operatorname{trop}(X)$, i.e. $\operatorname{trop}(X)$ is invariant under tropical scalar multiplication. So $\operatorname{trop}(X)$ lives in the *tropical projective torus* $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$.

From Proposition 2.24 we can see that $\operatorname{trop}(X)$ depends only on the matroid of $\mathcal{A}$, it does not depend on how the matroid is realized. It is referred to as a *tropicalized linear space*.

Definition 2.10. A *tropicalized linear space* over $\mathbb{K}$ is a tropical variety of the form $\operatorname{trop}(X)$ where X is a linear space in $T^n_{\mathbb{K}} \cong (\mathbb{K}^*)^{n+1}/\mathbb{K}^*$. That is, X is cut out by homogeneous linear forms in $\mathbb{K}[x_0^{\pm 1}, \dots, x_n^{\pm 1}]$.

We can turn the description of $\operatorname{trop}(X)$ in Proposition 2.24 into a definition and drop the representation as a linear ideal I, to generalize the construction to any matroid M, representable or not, and call the associated $\operatorname{trop}(M)$ a *tropical linear space.*

Definition 2.11. Let M be a matroid on $E = \{0, \dots, n\}$. The *tropical linear space* associated to M is

$$\operatorname{trop}(M) = \left\{ \mathbf{w} \in \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1} \,\middle|\, \min_{i \in C} w_i \text{ is attained at least twice } \forall C \in \mathcal{C}(M) \right\}.$$

Note that the definition of tropical linear spaces can be generalized to non-trivial valuations via Dressians.

Definition 2.12. Let M be a matroid on $E = \{0, \dots, n\}$. For any $\mathbf{w} \in \mathbb{R}^{n+1}$, The *initial matroid* $M_{\mathbf{w}}$ is a matroid on E whose circuits are the inclusion-minimal sets under $\{j \in C : w_j = \min_{i \in C} w_i\}$, where C runs over all circuits of M.

It can be shown that the circuits of $M_{\mathbf{w}}$ satisfy the axioms so $M_{\mathbf{w}}$ is indeed a matroid. Recall that the matroid polytope $\mathcal{P}(M)$ of M is the convex hull of the indicator vectors of all bases of M. Here the negative of the normal fan of a polytope is called the *outer normal fan.* Since each circuit of $M_{\mathbf{w}}$ is a subset of a circuit of M, each independent set of $M_{\mathbf{w}}$ is also independent in M. As optimization on matroids corresponds to linear programming on matroid polytopes, $\mathcal{P}(M_{\mathbf{w}})$ is a face of $\mathcal{P}(M)$.

Proposition 2.25. *For any $\mathbf{w} \in \mathbb{R}^{n+1}$, $\mathcal{P}(M_{\mathbf{w}})$ is the face of the matroid polytope $\mathcal{P}(M)$ at which w is maximized. Thus M_w is constant on the relative interior of cones in the outer normal fan of $\mathcal{P}(M)$.*

Now we can rewrite Definition 2.11. A vector $\mathbf{w}$ lies in $\operatorname{trop}(M)$ iff $\min_{i \in C} w_i$ is achieved at least twice for all circuits C of M, iff all circuits of $M_{\mathbf{w}}$ have size at least two, iff $M_{\mathbf{w}}$ has no loops. Therefore we have the following description of $\operatorname{trop}(M)$.

Theorem 2.26. *The tropical linear space* $\operatorname{trop}(M)$ *is the union of those cones of the outer normal fan of $\mathcal{P}(M)$ for which $M_{\mathbf{w}}$ has no loops:*

$$\operatorname{trop}(M) = \{\mathbf{w} \in \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1} : M_{\mathbf{w}} \textit{ has no loops}\}$$

This fan structure on trop(M) defined above is a subfan of the outer normal fan of the matroid polytope $\mathcal{P}(M)$, thus it is the coarsest possible fan structure on trop(M).

Now we describe a fan structure on trop(M) that is natural from a combinatorial perspective. Let M be a matroid on $\{0,\ldots,n\}$ and $\mathcal{L}_M$ be the lattice of flats of M. A flat F of M is represented by its indicator vector $\mathbf{e}_F = \sum_{i\in F}\mathbf{e}_i$. We regard $\mathbf{e}_F$ as an element in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$. For any chain of flats $\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_r \subsetneq E$, consider the polyhedral cone spanned by their incidence vectors

$$\sigma = \operatorname{cone}(\mathbf{e}_{F_1},\ldots,\mathbf{e}_{F_r}) + \mathbb{R}\mathbf{1} = \{\lambda_0\mathbf{1} + \lambda_1\mathbf{e}_{F_1} + \cdots + \lambda_r\mathbf{e}_{F_r} : \lambda_1,\ldots\lambda_r \geq 0\}\,.$$

Theorem 2.27. *Let M be a matroid on $E = \{0,\ldots,n\}$. The collection of cones* $\operatorname{cone}(\mathbf{e}_{F_1},\ldots,\mathbf{e}_{F_r}) + \mathbb{R}\mathbf{1}$*, where $\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_r \subsetneq E$ runs over all chains of flats of M, forms a pure simplicial fan of dimension* $\operatorname{rk}(M)-1$ *in* $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$*. The support of this fan equals the tropical linear space* trop(M).

In [AK06], the fan structures described in Theorem 2.26 and Theorem 2.27 are called the *coarse subdivision* and the *fine subdivision* of trop(M), respectively. In this paper the latter one is called the Bergman fan of M.

Definition 2.13. Let M be a matroid on $E = \{0,\ldots,n\}$. The *Bergman fan* Σ_M of M is a fan in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ consisting of cones $\sigma_{\mathcal{F}} := \operatorname{cone}(\mathbf{u}_{F_1},\ldots,\mathbf{u}_{F_k})$ for every flag of nonempty proper flats $\mathcal{F} = (F_1 \subsetneq \ldots \subsetneq F_k)$ in M, where $\mathbf{u}_F$ is the vector in the quotient space $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ corresponding to $\mathbf{e}_F$.

Consider the uniform matroid $U_{|E|,E}$ on E of full rank. Its lattice of flats $\mathcal{L}_{U_{|E|,E}}$ is the Boolean lattice on E. For every matroid M on E, $\mathcal{L}_M$ can be embedded on $\mathcal{L}_{U_{|E|,E}}$. Therefore, the Bergman fan $\Sigma_{U_{|E|,E}}$ of $U_{|E|,E}$ is a complete fan in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ and the Bergman fan Σ_M of every matroid M on E is a subfan of $\Sigma_{U_{|E|,E}}$. We will see that this fan is the normal fan of the permutohedron Π_n and is called the permutohedral fan Σ_{A_n}.

Definition 2.14. The n-dimensional *permutohedral fan* is the complete fan Σ_{A_n} whose d-dimensional cones are of the form $\sigma_{\mathcal{F}} = \operatorname{cone}(\mathbf{u}_{F_1},\ldots,\mathbf{u}_{F_d})$ for every chain $\mathcal{F} = (\emptyset \subsetneq F_1 \subsetneq \ldots \subsetneq F_d \subsetneq E)$ of nonempty proper subsets of E.

Recall that the *permutohedron* (or *permutahedron* in [Zie12]) Π_n is

$$\begin{aligned}\Pi_n &= \operatorname{conv}\{(x_0,\dots,x_n)\,|\,x_0\cdots x_n \in \mathfrak{S}_E\}\\ &= \left\{\mathbf{x}\in\mathbb{R}^{n+1}\,\middle|\,\sum_{i=0}^n x_i = \binom{n+1}{2}, \forall S\subseteq E : \sum_{i\in S} x_i \le \frac{|S|}{2}\right\}\\ &= \frac{n}{2}\mathbf{1} + \sum_{0\le i<j\le n}\left[-\frac{\mathbf{e}_j-\mathbf{e}_i}{2}, \frac{\mathbf{e}_j-\mathbf{e}_i}{2}\right].\end{aligned}$$

Π_n is an n-dimensional polytope contained in the hyperplane $\sum_{i=0}^n x_i = n(n+1)/2$ in $\mathbb{R}^{n+1}$. All $(n+1)!$ points in the convex hull representation above are vertices of Π_n because the symmetric group $\mathfrak{S}_E$ on E acts on Π_n by permuting the coordinates. The $(n-d)$-dimensional faces of Π_n correspond bijectively to a flag of nonempty proper subsets of E. For a flag $\mathcal{F} = (\emptyset \subsetneq F_1 \subsetneq \ldots \subsetneq F_d \subsetneq E)$ there is a face

$$\Pi_{\mathcal{F}} = \operatorname{conv}\{(x_0,\dots,x_n)\in \operatorname{vert}\Pi_n\,|\,\forall j\in[d]\,\forall k\in F_j\backslash F_{j-1}\,\forall l\in F_{j+1}\backslash F_j : x_k < x_l\}$$

corresponding to $\mathcal{F}$. The coordinates of every vertex of the face $\Pi_{\mathcal{F}}$ in positions in F_i are $0,\dots,|F_i|-1$, therefore the difference between any two vertices of $\Pi_{\mathcal{F}}$ is orthogonal to $\mathbf{e}_{F_i}$, for any $i\in[d]$. Thus, the permutohedral fan Σ_{A_n} is the normal fan of the permutohedron Π_n. We remark that the number of $(n-d)$-dimensional faces in Π_n is the number of ordered partitions of E into $d+1$ parts $F_1, F_2\backslash F_1,\dots,F_d\backslash F_{d-1}, E\backslash F_d$, namely $(d+1)!\left\{{n+1 \atop d+1}\right\}$.

Chapter 3

Chow Rings

3.1 Minkowski Weights and Chow Rings

3.1.1 Homology and cohomology

Let N be a lattice of rank n. We denote by $N^\vee$ be the dual lattice of N, and by $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$. For a rational fan Σ in $N_\mathbb{R}$, the set of k-dimensional cones in Σ is denoted by $\Sigma(k)$. For a ray $\rho \in \Sigma(1)$, write $\mathbf{u}_\rho$ for the *primitive ray vector* that generates $\rho \cap N$.
A fan Σ is called *complete* if its support $|\Sigma| = N_\mathbb{R}$. A fan Σ is said to be *smooth* if for all cones σ of Σ, the set of primitive ray vectors of σ can be extended to a basis of N, and *simplicial* if every k-dimensional cone is generated by k rays. As a smooth fan Σ is simplicial, it defines a simplicial complex on the set $\Sigma(1)$ of rays of Σ whose facets are the maximal cones of Σ, thus we can define the *star* of a cone σ in Σ to be

$$\operatorname{star}(\sigma, \Sigma) := \{\sigma' \in \Sigma \,|\, \sigma, \sigma' \subseteq \tau \text{ for some } \tau \in \Sigma\}$$

and the *link* of a cone σ in Σ to be

$$\operatorname{link}(\sigma, \Sigma) := \{\sigma' \in \Sigma \,|\, \sigma, \sigma' \subseteq \tau \text{ for some } \tau \in \Sigma, \text{ and } \sigma \cap \sigma' = \{0\}\}$$

which are subfans of the fan Σ.

Definition 3.1. Let Σ be a smooth fan in an n-dimensional latticed vector space $N_\mathbb{R}$. The *Chow ring* $A^\bullet(\Sigma)$ of Σ is the graded algebra

$$A^\bullet(\Sigma) = \mathbb{R}[x_\rho : \rho \in \Sigma(1)]/(I_\Sigma + J_\Sigma)$$

where I_Σ and J_Σ are the ideals of $\mathbb{R}[x_\rho : \rho \in \Sigma(1)]$ defined by

$$I_\Sigma := \left\langle \prod_{\rho \in S} x_\rho \;\middle|\; \text{the rays } S \subseteq \Sigma(1) \text{ do not form a cone in } \Sigma \right\rangle,$$

$$J_\Sigma := \left\langle \sum_{\rho \in \Sigma(1)} \langle \mathbf{u}_\rho, m \rangle x_\rho \;\middle|\; m \in N^\vee \right\rangle.$$

Elements in $A^1(\Sigma)$ are called *divisors* on Σ.

The ideal I_Σ is the Stanley-Reisner ideal of the simplicial complex defined by the smooth fan Σ. See [Sta07; MS04] for the interaction between combinatorics and commutative algebra.

Remark. The Chow ring $A^\bullet(\Sigma)$ takes coefficients initially in $\mathbb{Z}$, and is naturally isomorphic to the Chow ring $A^\bullet(X_\Sigma)$ of the smooth toric variety $X_\Sigma = \bigcup_{\sigma \in \Sigma} \operatorname{Spec} \mathbb{K}[\Sigma^\vee \cap N^\vee]$ associated to Σ by

$$A^\bullet(\Sigma) \xrightarrow{\sim} A^\bullet(X_\Sigma), \quad x_\sigma \mapsto [X_{\operatorname{star}(\sigma,\Sigma)}].$$

If Σ is a complete smooth fan, then $A^\bullet(\Sigma) \cong A^\bullet(X_\Sigma) \cong H^\bullet(X_\Sigma, \mathbb{Z})$, where $H^\bullet(X_\Sigma, \mathbb{Z})$ is the cohomology ring with coefficients in $\mathbb{Z}$, for which the Poincaré duality property is satisfied. This also holds with coefficients in $\mathbb{Q}$ instead of $\mathbb{Z}$ when Σ is simplicial instead of smooth, see [CLS11, §12.5].

Let $Z^k(\Sigma)$ be the $\mathbb{R}$-subspace of $\mathbb{R}[x_\rho : \rho \in \Sigma(1)]$ spanned by the square-free monomials

$$x_\sigma := \prod_{\rho \in \sigma} x_\rho$$

for all $\sigma \in \Sigma(k)$. That is,

$$Z^k(\Sigma) := \bigoplus_{\sigma \in \Sigma(k)} \mathbb{R} x_\sigma.$$

Proposition 3.1. *The degree k part $A^k(\Sigma)$ of the Chow ring of Σ is spanned by $Z^k(\Sigma)$ for each non-negative integer k. In particular, if $k > \dim \Sigma$, then $A^k(\Sigma) = 0$.*

Proof. Let σ be a cone in Σ, let $\rho_1, \rho_2, \ldots, \rho_l$ be its rays, and consider a degree k monomial of the form

$$x_{\rho_1}^{k_1} x_{\rho_2}^{k_2} \cdots x_{\rho_l}^{k_l}, \qquad k_1 \geq k_2 \geq \cdots \geq k_l \geq 1.$$

We show that the image of this monomial in $A^k(\Sigma)$ is in the span of $Z^k(\Sigma)$. We do this by descending induction on the dimension of σ. If $\dim \sigma = k$, there is nothing to prove. If otherwise, we use the smoothness of σ to choose $m \in N^\vee$ such that

$$\langle \mathbf{u}_{\rho_1}, m\rangle = -1 \quad \text{and} \quad \langle \mathbf{u}_{\rho_2}, m\rangle = \cdots = \langle \mathbf{u}_{\rho_l}, m\rangle = 0.$$

This shows that, modulo the relations given by I_Σ and J_Σ, we have

$$\begin{aligned} x_{\rho_1}^{k_1} x_{\rho_2}^{k_2} \cdots x_{\rho_l}^{k_l} &= x_{\rho_1}^{k_1-1} x_{\rho_2}^{k_2} \cdots x_{\rho_l}^{k_l} \sum_{\rho \in \Sigma(1) \setminus \{\rho_1\}} \langle \mathbf{u}_\rho, m\rangle x_\rho \\ &= x_{\rho_1}^{k_1-1} x_{\rho_2}^{k_2} \cdots x_{\rho_l}^{k_l} \sum_{\rho \in \mathrm{link}(\sigma,\Sigma)(1)} \langle \mathbf{u}_\rho, m\rangle x_\rho . \end{aligned}$$

The induction hypothesis applies to each of the terms in the expansion of the right-hand side. □

Definition 3.2. An l-dimensional *Minkowski weight* $\Delta \in \mathrm{MW}_l(\Sigma)$ is a function $\Delta : \Sigma(l) \to \mathbb{R}$ such that for each $\tau \in \Sigma(l-1)$, the function Δ satisfies the balancing condition

$$\sum_{\sigma \supseteq \tau} \Delta(\sigma) \mathbf{u}_{\sigma\setminus\tau} \in \mathrm{span}_{\mathbb{R}}(\tau)$$

where $\sigma\setminus\tau$ denotes the unique ray of an l-dimensional cone σ that is not in τ.

The group of l-dimensional weights on Σ can be identified with the dual of $Z^l(\Sigma)$ under the tautological isomorphism

$$t_\Sigma : \mathbb{R}^{\Sigma_l} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}}(Z^l(\Sigma), \mathbb{R}), \qquad \Delta \mapsto (x_\sigma \mapsto \Delta(\sigma)) .$$

By dualizing the quotient map $Z^l(\Sigma) \twoheadrightarrow Z^l(\Sigma)/\left(Z^l(\Sigma) \cap (I_\Sigma + J_\Sigma)\right) \cong A^l(\Sigma)$ in Proposition 3.1, the target of t_Σ contains $\mathrm{Hom}_{\mathbb{R}}(A^l(\Sigma), \mathbb{R})$ as a subgroup.

Theorem 3.2. *The isomorphism t_Σ restricts to the isomorphism*

$$\mathrm{MW}_l(\Sigma) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}}(A^l(\Sigma), \mathbb{R})$$

Proof. The homomorphisms from $A^l(\Sigma)$ to $\mathbb{R}$ bijectively correspond to the homomorphisms from $Z^l(\Sigma)$ to $\mathbb{R}$ that vanish on $Z^l(\Sigma) \cap (I_\Sigma + J_\Sigma) \subseteq Z^l(\Sigma)$, which is a subspace spanned by polynomials of the form

$$\left(\sum_{\rho \in \mathrm{link}(\tau,\Sigma)(1)} \langle \mathbf{u}_\rho, m\rangle x_\rho \right) x_\tau \tag{3.1}$$

where $\tau \in \Sigma(l-1)$ and m is an element perpendicular to $\langle\tau\rangle$. It follows that an l-dimensional weight Δ corresponds to a homomorphism $A^l(\Sigma) \to \mathbb{R}$ iff

$$\sum_{\tau\subseteq\sigma\in\Sigma(l)} \Delta(\sigma)\langle \mathbf{u}_{\sigma\setminus\tau}, m\rangle = 0 \text{ for all } m \in \langle\tau\rangle^{\perp},$$

which is equivalent to the balancing condition on Δ at τ since $\langle\tau\rangle^{\perp\perp} = \langle\tau\rangle$.

To prove that the polynomials (3.1) span $Z^l(\Sigma) \cap (I_\Sigma + J_\Sigma)$, first, those polynomials are generated by $\{x_\sigma \mid \sigma \in \Sigma(l)\}$ so they belong to $Z^l(\Sigma)$, and as $m \in \langle\tau\rangle^\vee$,

$$\left(\sum_{\rho\in\mathrm{link}(\tau,\Sigma)(1)} \langle \mathbf{u}_\rho, m\rangle x_\rho\right) x_\tau = \left(\sum_{\rho\in\Sigma(1)} \langle \mathbf{u}_\rho, m\rangle x_\rho\right) x_\tau - \left(\sum_{\rho\notin\mathrm{star}(\tau,\Sigma)} \langle \mathbf{u}_\rho, m\rangle x_\rho\right) x_\tau,$$

where the two terms of the right hand side belong to J_Σ and I_Σ, respectively. Now let

$$a_J = \sum_{m\in N^\vee} p_m \sum_{\rho\in\Sigma(1)} \langle \mathbf{u}_\rho, m\rangle x_\rho$$

be any element in J_Σ such that $a_J + a_I \in Z^l(\Sigma)$ for some $a_I \in I_\Sigma$. By choosing a suitable a_I we can assume that $p_m = \sum_{\tau\in\Sigma(l-1)} \alpha_{\tau,m} x_\tau$, $\alpha_{\tau,m} \in \mathbb{R}$. So the corresponding element in $Z^l(\Sigma)$ is a square-free polynomial

$$\sum_{m\in N^\vee} \sum_{\tau\in\Sigma(l-1)} \alpha_{\tau,m} x_\tau \sum_{\rho\in\mathrm{star}(\tau,\Sigma)(1)} \langle \mathbf{u}_\rho, m\rangle x_\rho,$$

where $\langle \mathbf{u}_\rho, m\rangle$ must be zero for any ray ρ in τ. This is a linear combination of polynomials (3.1). □

The isomorphism

$$t_\Sigma : \mathrm{MW}_l(\Sigma) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}}(A^l(\Sigma), \mathbb{R}), \quad \Delta \mapsto (x_\sigma \mapsto \Delta(\sigma))$$

in Theorem 3.2 is an analogue of the Kronecker duality map in algebraic topology that relates cohomology elements to homology elements. This isomorphism allows one to define the *cap product* by

$$A^k(\Sigma) \times \mathrm{MW}_l(\Sigma) \to \mathrm{MW}_{l-k}(\Sigma), \quad (\xi, \Delta) \mapsto \xi \cap \Delta := (\sigma \mapsto (t_\Sigma\Delta)(\xi \cdot x_\sigma)),$$

which makes $\mathrm{MW}_\bullet(\Sigma)$ a graded $A^\bullet(\Sigma)$-module.

If Σ is complete, then a d-dimensional weight satisfies the balancing condition iff it is constant. Therefore, in this case, $\mathrm{MW}_d(\Sigma) \cong \mathbb{R}$. We will see in Proposition 3.7 that the Bergman fan of a matroid has the same property. If the fan Σ satisfies $\mathrm{MW}_d(\Sigma) \cong \mathbb{R}$, then one can define the *fundamental class* Δ_Σ to be its generator (unique up to scaling), and we have maps

$$\delta : A^\bullet(\Sigma) \to \mathrm{MW}_{d-\bullet}(\Sigma), \quad \xi \mapsto \xi \cap \Delta_\Sigma.$$

In particular, noting that $\mathrm{MW}_0(\Sigma) = \mathbb{R}$, we have the *degree map*

$$\int_\Sigma : A^d(\Sigma) \to \mathbb{R}, \quad \xi \mapsto \xi \cap \Delta_\Sigma.$$

The construction of Chow ring and Minkowski weights is functorial in the following sense:

Definition 3.3. An inclusion of fans $\iota : \Sigma' \hookrightarrow \Sigma$ defines the *pullback map* ι^*, which is a surjective map of graded $\mathbb{R}$-algebras defined by

$$\iota^* : A^\bullet(\Sigma) \twoheadrightarrow A^\bullet(\Sigma'), \quad x_\rho \mapsto \begin{cases} x_\rho & \text{if } \rho \in \Sigma(1) \cap \Sigma'(1) \\ 0 & \text{otherwise.} \end{cases}$$

Dualizing the pullback map ι^* gives us the *pushforward map*

$$\iota_* : \mathrm{MW}_\bullet(\Sigma') \hookrightarrow \mathrm{MW}_\bullet(\Sigma), \quad \Delta' \mapsto (\sigma \mapsto \Delta'(\sigma) \text{ for } \sigma \in \Sigma' \text{ and } 0 \text{ otherwise}).$$

3.1.2 Complete smooth fans

Because we can get the Chow ring and Minkowski weights of a smooth fan as pullback respective pushforward of those of a complete fan, we consider the complete smooth fans in the remaining part of this section.

A divisor $D = \sum_{\rho\in\Sigma(1)} c_\rho x_\rho \in A^1(\Sigma)$ defines a piecewise linear function φ_D that is linear on each cone of Σ by setting $\varphi_D(\mathbf{u}_\rho) = c_\rho$. A divisor D is *nef* if φ_D is a convex function on $N_\mathbb{R}$, i.e. $\varphi_D(\mathbf{u}) + \varphi_D(\mathbf{u}') \geq \varphi_D(\mathbf{u}+\mathbf{u}')$ for any $\mathbf{u}, \mathbf{u}' \in N_\mathbb{R}$. If further the inequalities $\varphi_D(\mathbf{u}) + \varphi_D(\mathbf{u}') \geq \varphi_D(\mathbf{u}+\mathbf{u}')$ are strict whenever $\mathbf{u}, \mathbf{u}'$ are not in a common cone of Σ, we say that D is *ample.* As conical combinations of nef (ample) divisors are nef (ample), they form a cone $\overline{\mathfrak{K}}_\Sigma$ ($\mathfrak{K}_\Sigma$) in $A^1(\Sigma)$ called the *nef (ample) cone* of Σ. It is easy to check that $\overline{\mathfrak{K}}_\Sigma$ is the closure of $\mathfrak{K}_\Sigma$ and $\mathfrak{K}_\Sigma$ is the interior of $\overline{\mathfrak{K}}_\Sigma$. Nef divisors of Σ correspond to a certain family of polytopes called *deformations* of Σ, which are polytopes Q in $N_\mathbb{R}^\vee$ whose outer normal fans Σ_Q coarsen Σ. We denote by $\mathrm{Def}(\Sigma)$ the set of deformations of Σ.

Theorem 3.3. *Let Σ be a smooth complete fan.*

1. *A nef divisor $D = \sum_{\rho\in\Sigma(1)} c_\rho x_\rho$ on Σ defines a deformation P_D by*

$$P_D := \{m \in N_{\mathbb{R}}^\vee \mid \langle \mathbf{u}_\rho, m\rangle \leq c_\rho \ \forall \rho \in \Sigma(1)\}$$

whereas a deformation $P \subset N_{\mathbb{R}}^\vee$ of Σ defines a nef divisor

$$D_P := \sum_{\rho\in\Sigma(1)} \max_{m\in P}\{\langle \mathbf{u}_\rho, m\rangle\} x_\rho.$$

2. *Two nef divisors $D = \sum_{\rho\in\Sigma(1)} c_\rho x_\rho$ and $D' = \sum_{\rho\in\Sigma(1)} c'_\rho x_\rho$ are equal in $A^1(\Sigma)$ iff P_D and $P_{D'}$ are parallel translates, and moreover, $P_{D+D'} = P_D + P_{D'}$.*

Proof. Let $D = \sum_{\rho\in\Sigma(1)} c_\rho x_\rho$ be a nef divisor on Σ. We want to show that P_D is a deformation of Σ. P_D is bounded because of the smoothness and completeness of Σ. The set of outer normal rays of facets is a subset $\Sigma(1)$. What is left to show is $P_D \neq \emptyset$, i.e., the outer normal fan of P_D is complete. Divisor D is nef iff φ_D is a convex function on $N_{\mathbb{R}}$, iff $Q := \{(\mathbf{x}, y) \in N_{\mathbb{R}} \times \mathbb{R} \mid \mathbf{x} \in N_{\mathbb{R}}, y \geq \varphi_D(\mathbf{x})\}$ is a convex set. It is known that $(0,0) \in \partial Q$ as $\varphi_D(0) = 0$. Apply the isolation theorem on the interior of Q and $(0,0)$, we know that there exists some $w \in N_{\mathbb{R}}^\vee$ such that $\langle(\mathbf{x}, y), (w, -1)\rangle = \langle \mathbf{x}, w\rangle - y \leq 0$ for all $(\mathbf{x}, y) \in Q$. Such w must be in P_D as $\langle \mathbf{u}_\rho, w\rangle \leq \varphi_D(\mathbf{u}_\rho) = c_\rho$.
Now let $P \subset N_{\mathbb{R}}^\vee$ be a deformation of Σ. Define $c_\rho := \max\{\langle \mathbf{u}_\rho, m\rangle \mid m \in P\}$. The maximums exist because for any $\rho \in \Sigma(1)$, $m \mapsto \langle \mathbf{u}_\rho, m\rangle$ is a linear $\mathbb{R}$-function, which is continuous, and the polytope P is a compact set. For a point $\mathbf{u} = \sum_{i=1}^n \alpha_i \mathbf{u}_{\sigma_i} \in \sigma \subset N_{\mathbb{R}}$, where $\mathbf{u}_{\sigma_i}$ are the primitive ray generators of $\sigma \in \Sigma(n)$ and $\alpha_i \geq 0$ for all i, we have

$$\varphi_D(\mathbf{u}) = \sum_{i=1}^n \alpha_i \max_{m\in P}\langle \mathbf{u}_{\sigma_i}, m\rangle = \max_{m\in P}\left\langle \sum_{i=1}^n \alpha_i \mathbf{u}_{\sigma_i}, m\right\rangle = \max_{m\in P}\langle \mathbf{u}, m\rangle,$$

where the second equality follows from the fact that P is a deformation of Σ, so the maximum is attained by the vertex corresponding to the cone of the outer normal fan of P containing σ. We check the convexity of φ_D:

$$\begin{aligned}\varphi_D(\mathbf{u} + \mathbf{u}') &= \max_{m\in P}\langle \mathbf{u} + \mathbf{u}', m\rangle = \max_{m\in P}(\langle \mathbf{u}, m\rangle + \langle \mathbf{u}', m\rangle) \\ &\leq \max_{m\in P}\langle \mathbf{u}, m\rangle + \max_{m\in P}\langle \mathbf{u}', m\rangle = \varphi_D(\mathbf{u}) + \varphi_D(\mathbf{u}'),\end{aligned}$$

for any $\mathbf{u}, \mathbf{u}' \in N_{\mathbb{R}}$. Therefore such c_ρ define a nef divisor D on Σ.
Now we assume $D = \sum_\rho c_\rho x_\rho$ and $D' = \sum_\rho c'_\rho x_\rho$ to be equal in $A^1(\Sigma)$. That is equivalent to $D - D' \in \left\langle \sum_{\rho\in\Sigma(1)} \langle \mathbf{u}_\rho, m\rangle x_\rho \,\middle|\, m \in N^\vee \right\rangle$, that is,

$$D-D' = \sum_{\rho\in\Sigma(1)} (c_\rho - c'_\rho) x_\rho = \sum_{m\in N^\vee} \alpha_m \sum_{\rho\in\Sigma(1)} \langle \mathbf{u}_\rho, m\rangle x_\rho = \sum_{\rho\in\Sigma(1)} \left\langle \mathbf{u}_\rho, \sum_{m\in N^\vee} \alpha_m m \right\rangle x_\rho$$

for some $\alpha_m \in \mathbb{R}$ where all but finitely many α_m are zero. As $\sum_{m\in N^\vee} \alpha_m m \in N^\vee_{\mathbb{R}}$ could be any real linear combination of $N^\vee$, i.e., any element of $N^\vee_{\mathbb{R}}$, we have that $D = D'$ in $A^1(\Sigma)$ iff $c_\rho - c'_\rho = \langle \mathbf{u}_\rho, n\rangle$ for some $n \in N^\vee_{\mathbb{R}}$. That is,

$$\begin{aligned} P_D &= \left\{ m \in N^\vee_{\mathbb{R}} \mid \langle \mathbf{u}_\rho, m\rangle \le c'_\rho + \langle \mathbf{u}_\rho, n\rangle \ \forall \rho \in \Sigma(1) \right\} \\ &= \{ m \in N^\vee_{\mathbb{R}} \mid \langle \mathbf{u}_\rho, m - n\rangle \le c_\rho \ \forall \rho \in \Sigma(1) \} = P_{D'} + n \end{aligned}$$

for some $n \in N^\vee_{\mathbb{R}}$.
At last, $P_{D+D'} = P_D + P_{D'}$, $D = D_{P_D}$ and $P = P_{D_P}$ can be checked easily.

□

In other words, giving $\mathrm{Def}(\Sigma)$ a structure of a cone by Minkowski sums, we have a bijection

$$\overline{\mathfrak{K}}_\Sigma \xleftrightarrow{\sim} \mathrm{Def}(\Sigma)/\sim$$

via $D \mapsto P_D$ and $P \mapsto D_P$, where $P \sim P'$ iff $P = P' + n$ for some $n \in N^\vee_{\mathbb{R}}$.

Corollary 3.4. *Any nef divisor $D \in A^1(\Sigma)$ is* effective*, i.e., it can be written as a conical linear combination $D = \sum_{\rho\in\Sigma(1)} c_\rho x_\rho$ where $c_\rho \ge 0$ for all ρ. In particular, any ample divisor can be written as a positive linear combination of x_ρ, $\rho \in \Sigma(1)$.*

Proof. We can assume P_D with $0 \in \operatorname{relint} P_D$ by translating. The second statement follows from the fact that $\mathfrak{K}_\Sigma$ is the interior of $\overline{\mathfrak{K}}_\Sigma$. Remark that P_D is full-dimensional when D is ample. □

We have seen that a complete fan Σ satisfies $\mathrm{MW}_d(\mathbb{R}) \cong \mathbb{R}$. We take the fundamental class Δ_Σ to be $\Delta_\Sigma(\sigma) = 1 \ \forall \sigma \in \Sigma(n)$. The following main result of [FS97] is a tropical geometric analogue of the Poincaré duality property in algebraic topology.

Theorem 3.5. *For a smooth complete rational fan* Σ*, the cap product by* Δ_Σ

$$\delta : A^\bullet(\Sigma) \xrightarrow{\sim} \mathrm{MW}^\bullet(\Sigma) := \mathrm{MW}_{d-\bullet}(\Sigma), \quad \xi \mapsto \xi \cap \Delta_\Sigma$$

is an isomorphism as graded rings where $\mathrm{MW}^\bullet(\Sigma)$ *is given a ring structure by stable intersection of tropical cycles.*

3.2 Chow Rings of Matroids

3.2.1 Chow rings of Bergman fans

For a loopless matroid M of rank $r = d + 1$ on a ground set E, recall that the Bergman fan Σ_M of M is the pure d-dimensional smooth rational fan in $(\mathbb{Z}^E/\mathbb{Z}\mathbf{1})_\mathbb{R}$ consisting of cones $\sigma_\mathcal{F} := \mathrm{cone}(\mathbf{u}_{F_1}, \dots, \mathbf{u}_{F_k})$ for every flag of nonempty proper flats $\mathcal{F} = (F_1 \subsetneq \dots \subsetneq F_k)$ in M, where $\mathbf{u}_F$ is the vector in the quotient space $\mathbb{R}^E/\mathbb{R}\mathbf{1}$ corresponding to $\mathbf{e}_F = \sum_{i \in F} \mathbf{e}_i \in \mathbb{R}^E$.

Definition 3.4. The *Chow ring* $A^\bullet(M)$ of a loopless matroid M is the Chow ring $A^\bullet(\Sigma_M)$ of its Bergman fan Σ_M. Explicitly,

$$A^\bullet(M) = \frac{\mathbb{R}[x_F : F \in \mathcal{L}_M \backslash \{\emptyset, E\}]}{\langle x_F x_{F'} \mid F, F' \text{ incomparable}\rangle + \left\langle \sum_{F \supseteq a} x_F - \sum_{G \supseteq b} x_G \;\middle|\; a, b \in \mathfrak{A}(M) \right\rangle}.$$

We call elements of $A^1(M)$ *divisors* on M, and denote by $\alpha := \sum_{F \supseteq a} x_F$ the *hyperplane class* of M for any $a \in \mathfrak{A}(M)$.

The Chow ring of a loopless matroid M was first studied in [FY04] in a slightly different presentation

$$A^\bullet_{\mathrm{FY}}(M) := \frac{\mathbb{R}[z_F : F \in \mathcal{L}_M \backslash \{\emptyset\}]}{\langle z_F z_{F'} \mid F, F' \text{ incomparable}\rangle + \left\langle \sum_{F \supseteq a} z_F \;\middle|\; a \in \mathfrak{A}(M) \right\rangle}.$$

That is, we have $x_F = z_F$ for every nonempty proper flat $F \in \mathcal{L}_M$, and $z_E = -\alpha$. In this thesis, we always use the variable names x_F for elements in $A^\bullet(M)$, and z_F for elements in $A^\bullet_{\mathrm{FY}}(M)$. For instance, in the summation $\sum_{F \supseteq a} x_F$ we assume $F \subsetneq E$.

Remark. If M is representable and realized as a hyperplane arrangement $\mathcal{A}$, the wonderful compactification $\overline{Y}$ of the complement Y of $\mathcal{A}$ is a tropical compactification of Y in the toric variety X_{Σ_M}. It was shown in [FY04] that the inclusion $\overline{Y} \hookrightarrow X_{\Sigma_M}$ induces a Chow equivalence $A^\bullet(\overline{Y}) \cong A^\bullet(X_{\Sigma_M}) \cong$

$A^\bullet(M)$. See [MS15; Den14; Fei05] for tropical and wonderful compactifications.

The Minkowski weights on a complete fan Σ satisfy $\mathrm{MW}_d(\Sigma_M) \cong \mathbb{R}$. We will show that the Bergman fans of matroids have the same property. The following Lemma is a consequence of the shellability of Σ_M [Bjö92].

Lemma 3.6. *The Bergman fan Σ_M is connected in codimension 1.*

Proof. We will show that for any two d-dimensional cones $\sigma_{\mathcal{F}}$, $\sigma_{\mathcal{G}}$ in Σ_M, there is a sequence

$$\sigma_{\mathcal{F}} = \sigma_0 \supset \tau_1 \subset \sigma_1 \supset \cdots \subset \sigma_{l-1} \supset \tau_l \subset \sigma_l = \sigma_{\mathcal{G}},$$

where τ_i is a common facet of σ_{i-1} and σ_i in Σ_M. We express this by writing $\sigma_{\mathcal{F}} \sim \sigma_{\mathcal{G}}$.
We prove this by induction on the dimension d of Σ_M. If $\min \mathcal{F} = \min \mathcal{G}$, then the induction hypothesis applied to $M/\min \mathcal{F}$ shows that $\sigma_{\mathcal{F}} \sim \sigma_{\mathcal{G}}$. If otherwise, we choose a flag of nonempty proper flats $\mathcal{H}$ maximal among those satisfying $\min \mathcal{F} \cup \min \mathcal{G} \subseteq \min \mathcal{H}$. By the induction hypothesis applied to $M/\min \mathcal{F}$ and $M/\min \mathcal{G}$ respectively, we have

$$\sigma_{\mathcal{F}} \sim \sigma_{\{\min \mathcal{F}\} \cup \mathcal{H}} \quad \text{and} \quad \sigma_{\mathcal{G}} \sim \sigma_{\{\min \mathcal{G}\} \cup \mathcal{H}},$$

Hence $\sigma_{\mathcal{F}} \sim \sigma_{\mathcal{G}}$. Since any 1-dimensional fan is connected in codimension 1, this completes the induction. □

Proposition 3.7. *A d-dimensional weight on Σ_M satisfies the balancing condition iff it is constant. Furthermore,* $\mathrm{MW}_d(\Sigma_M) \cong \mathbb{R}$.

Proof. The proof is based on the flat partition property (F2) for matroids M on E: If F is a flat of M, then the flats of M that cover F partition $E \backslash F$.
Let $\tau_{\mathcal{G}}$ be a codimension 1 cone in the Bergman fan Σ_M where $\mathcal{G} = (\emptyset \subsetneq G_1 \subsetneq \cdots \subsetneq G_{d-1} \subsetneq E)$ is a flag of nonempty proper flats. Set $G_0 := \emptyset$ and $G_d := E$. Let $F_1, \ldots, F_m$ be the nonempty proper flats of M such that $(G_0 \subsetneq \cdots \subsetneq G_{l-1} \subsetneq F_j \subsetneq G_l \subsetneq \cdots \subsetneq G_d)$ is a full flag for each $j = 1, \ldots, m$. The flat partition property is equivalent to

$$\sum_{j=1}^{m} \mathbf{u}_{F_j} = \mathbf{u}_{G_l} + (m-1)\mathbf{u}_{G_{l-1}}.$$

That means,

$$\sum_{\rho \in \mathrm{star}(\tau_{\mathcal{G}})(1)} \mathbf{u}_\rho = 0 \quad \text{in} \quad (\mathbb{R}^E/\mathbb{R}\mathbf{1})/\langle \tau_{\mathcal{G}} \rangle$$

and any proper subset of $\{\mathbf{u}_\rho \,|\, \rho \in \operatorname{star}(\tau_{\mathcal{G}})(1)\}$ is linearly independent. Therefore, a d-dimensional weight Δ on Σ_M satisfies the balancing condition at $\tau_{\mathcal{G}}$ iff Δ is constant on cones containing $\tau_{\mathcal{G}}$. By the connectedness of Lemma 3.6, the latter condition for every $\tau_{\mathcal{G}}$ implies that Δ is constant. □

The Bergman fan Σ_M of a loopless matroid M of rank $r = d + 1$ satisfies $\mathrm{MW}_d(\Sigma_M) \cong \mathbb{R}$, thus we can take the fundamental class, called the *Bergman class*, to be

$$\Delta_M \in \mathrm{MW}_d(\Sigma_M) \text{ where } \Delta_M(\sigma) = 1 \;\; \forall \sigma \in \Sigma_M(d),$$

and the *degree map*

$$\int_M : A^d(\Sigma_M) \to \mathbb{R}, \qquad \xi \mapsto \xi \cap \Delta_M$$

of M. Explicitly, this map is determined by

$$\int_M x_{F_1} x_{F_2} \cdots x_{F_d} = 1 \text{ for every maximal chain } F_1 \subsetneq \cdots \subsetneq F_d \text{ in } \mathcal{L}_M \backslash \{\emptyset, E\}.$$

Recall that the Bergman fan of the Boolean matroid $U_{|E|,E}$ on $E = \{0, \ldots, n\}$ is the permutohedral fan Σ_{A_n}, which is a complete fan in $\mathbb{R}^E/\mathbb{R}\mathbf{1}$ with primitive ray generators $\{\mathbf{u}_S : \emptyset \subsetneq S \subsetneq E\}$. Note that the dual lattice of $N = \mathbb{Z}^E/\mathbb{Z}\mathbf{1}$ is $N^\vee = \mathbf{1}^\perp := \left\{(y_0, \ldots, y_n) \in \mathbb{Z}^E \,\middle|\, \sum_{i=0}^n y_i = 0\right\}$. The following theorem is a specialization of Theorem 3.3, see also [AA17; BB11].

Theorem 3.8. *The following are equivalent for a divisor* $D = \sum_{\emptyset \subsetneq S \subsetneq E} c_S x_S \in A^1(\Sigma_{A_n})$*:*

1. *D is a nef divisor on Σ_{A_n},*
2. *the function $c_{(\cdot)} : 2^E \to \mathbb{R}$ satisfies the submodular property*
$$c_A + c_B \leq c_{A\cup B} + c_{A\cap B} \text{ for every } A, B \subseteq E \text{ where } c_\emptyset = c_E = 0,$$
3. *the polytope $P_D = \{m \in N^\vee_{\mathbb{R}} \,|\, \langle \mathbf{u}_S, m\rangle \leq c_S \; \forall \emptyset \subsetneq S \subsetneq E\}$ is a deformation of Σ_{A_n},*
4. *every edge of P_D is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i \neq j \in E$.*

Given a submodular function $c_{(\cdot)} : 2^E \to \mathbb{Z}$ with $c_\emptyset = 0$ but c_E possibly nonzero, the *generalized permutohedron* associated to $c_{(\cdot)}$ is the polytope

$$P(c) := \left\{ y \in (\mathbb{R}^E)^\vee \,\middle|\, \langle \mathbf{1}, y\rangle = c_E \text{ and } \langle \mathbf{e}_S, y\rangle \leq c_S \; \forall \emptyset \subsetneq S \subsetneq E \right\}.$$

This polytope lives in an affine translate of $N_{\mathbb{R}}^{\vee}$ in $\mathbb{R}^E$. We can translate a generalized permutahedron $P(c)$ to $N_{\mathbb{R}}^{\vee}$ to get the polytope P_D in Theorem 3.8 as follows: Fix an element $i \in E$, and consider

$$P(c)-c_E\mathbf{e}_i = \{m \in N_{\mathbb{R}}^{\vee} \mid \langle \mathbf{u}_S, m\rangle \le c_S - c_E \,\forall\, S \ni i \text{ and } \langle \mathbf{u}_S, m\rangle \le c_S \,\forall\, S \not\ni i\}\,.$$

In other words, the nef divisor in $A^1(\Sigma_{A_n})$ that the polytope $P(c)$ corresponds to is

$$-c_E\alpha + \sum_{\emptyset\subsetneq S\subsetneq E} c_S x_S = \sum_{\emptyset\subsetneq S\subseteq E} c_S z_S \in A^{\bullet}_{\mathrm{FY}}(\Sigma_{A_n}).$$

Now, for a loopless matroid M on $E = \{0, \dots, n\}$, we have an inclusion of fans $\iota_M : \Sigma_M \hookrightarrow \Sigma_{A_n}$, and the pushforward map $\iota_{M*} : \mathrm{MW}_{\bullet}(\Sigma_M) \hookrightarrow \mathrm{MW}_{\bullet}(\Sigma_{A_n})$ defines the Bergman class Δ_M of M as an element of $\mathrm{MW}_d(\Sigma_{A_n})$. Moreover, we have the pullback map

$$\iota_M^* : A^{\bullet}(\Sigma_{A_n}) \twoheadrightarrow A^{\bullet}(\Sigma_M), \qquad x_S \mapsto \begin{cases} x_S & \text{if } \emptyset \subsetneq S \subsetneq E \text{ is a flat of } M, \\ 0 & \text{otherwise.} \end{cases}$$

For $\emptyset \subsetneq S \subsetneq E$, we may need to clarify whether a variable x_S is an element of $A^{\bullet}(M)$ or $A^{\bullet}(\Sigma_{A_n})$. We thus often denote

$$x_S(M) := \iota_M^* x_S,$$

in which case x_S is considered as an element of $A^{\bullet}(\Sigma_{A_n})$ and $x_S(M) \in A^{\bullet}(M)$.

3.2.2 The simplicial presentation

Now we define the third presentation of the Chow ring of a matroid, called the simplicial presentation, which is introduced in [BES19]. This presentation is nothing else but an upper triangular linear change of variables in $A^{\bullet}(M)$. However, it has an excellent combinatorial interpretation that can be used to deduce the Poincaré duality property from that of the Chow ring of the permutohedral variety.

For a subset S of $E = \{0, \dots, n\}$, denote by

$$\nabla_S := \mathrm{conv}\{-\mathbf{e}_i : i \in S\} \subset \mathbb{R}^E$$

the negative standard simplex of S. Remark that

$$\begin{aligned}\nabla_S &= \left\{ \mathbf{x} \in \mathbb{R}^E \,\middle|\, \sum_{i \in S} x_i = -1,\ x_i = 0\, \forall i \notin S,\ x_i \le 0\, \forall i \right\} \\ &= \left\{ \mathbf{x} \in \mathbb{R}^E \,\middle|\, \mathbf{x} \cdot \mathbf{1} = -1,\ \mathbf{x} \cdot \mathbf{e}_T \le c_T \right\}\end{aligned}$$

where $c_T = -1$ for all $T \supseteq S$ and 0 otherwise. As the edges of ∇_S are parallel translates of $\mathbf{e}_i - \mathbf{e}_j$ for $i \neq j \in S$, Theorem 3.8 implies that ∇_S is a deformation of Σ_{A_n} whose corresponding nef divisor is

$$h_S := \alpha + \sum_{\emptyset \subsetneq T \subsetneq E} c_T x_T = \alpha - \sum_{S \subseteq T \subsetneq E} x_T = -\sum_{T \supseteq S} z_T \in A^\bullet_{\mathrm{FY}}(\Sigma_{A_n}).$$

Now we consider the presentation of $A^\bullet(M)$ given by pullbacks of these nef divisors of standard simplices. For M a loopless matroid on E and $\emptyset \neq S \subseteq E$, denote $h_S(M) := \iota_M^* h_S$. If $F \in \mathcal{L}_M$ is the closure of S, we have

$$h_S(M) := \iota_M^* h_S = -\sum_{T \supseteq S} z_T(M) = -\sum_{F \subseteq G \in \mathcal{L}_M} z_G(M) = \iota_M^* h_F,$$

as $z_T(M) = \iota_M^* h_T = 0$ for all $T \subseteq E$ not a flat of M. We will write simply h_F for $h_F(M)$ when there is no confusion.

Definition 3.5. For M a loopless matroid on E, the *simplicial presentation* $A^\bullet_\nabla(M)$ of the Chow ring of M is the presentation of $A^\bullet(M)$ whose generators are $\{h_F : F \in \mathcal{L}_M \backslash \{\emptyset\}\}$ where

$$h_F := -\sum_{G \supseteq F} z_G \in A^\bullet_{\mathrm{FY}}(M).$$

The linear change of variables from $\{z_F : F \in \mathcal{L}_M \backslash \{\emptyset\}\}$ to $\{h_F : F \in \mathcal{L}_M \backslash \{\emptyset\}\}$ is given by an upper triangular matrix, thus is invertible by the Möbius inversion

$$-z_F = \sum_{G \supseteq F} \mu(F, G) h_G$$

where μ is the Möbius function on the lattice $\mathcal{L}_M$. Thus, the explicit presentation of $A^\bullet_\nabla(M)$ is

$$A^\bullet_\nabla(M) := \mathbb{R}\left[h_F : F \in \mathcal{L}_M \backslash \{\emptyset\}\right] / (I + J)$$

where

$$I = \left\langle \left(\sum_{G \supseteq F} \mu(F, G) h_G \right) \left(\sum_{G' \supseteq F'} \mu(F', G') h_{G'} \right) : F, F' \text{ incomparable} \right\rangle$$

and

$$J = \langle h_a : a \in \mathfrak{A}(M) \rangle .$$

Denote by $\mathcal{L}_M^{\geq 2}$ the set of flats of M of rank at least 2. Noting that $h_a = 0 \in A^\bullet_\nabla(M)$ for any $a \in \mathfrak{A}(M)$. We define $\left\{h_F : F \in \mathcal{L}_M^{\geq 2}\right\}$ to be the *simplicial generators* of the Chow ring of M.

In [FY04], a Gröbner basis for the defining ideal of $A^\bullet_{\mathrm{FY}}(M)$ is given. Pick a total order on elements of $\mathcal{L}_M$ such that $F > G$ if $\mathrm{rk}_M(F) \leq \mathrm{rk}_M(G)$, and take the induced lex monomial order on $A^\bullet_{\mathrm{FY}}(M)$.

Theorem 3.9 ([FY04], Theorem 1)**.** *The following form a Gröbner basis for the ideal of $A^\bullet_{\mathrm{FY}}(M)$:*

$$\begin{cases} z_F z_G & F, G \text{ incomparable} \\ z_F \left(\sum_{H \supseteq G} z_H\right)^{\mathrm{rk}(G)-\mathrm{rk}(F)} & F \subsetneq G \\ \left(\sum_{H \supseteq G} z_H\right)^{\mathrm{rk}(G)} & G \in \mathcal{L}_M \backslash \{\emptyset\}. \end{cases}$$

This Gröbner basis computation carries over to the simplicial presentation easily.

Proposition 3.10. *The following is a Gröbner basis for the defining ideal of $A^\bullet_\nabla(M)$ with respect to the lex monomial ordering induced by $>$:*

$$\begin{cases} \left(\sum_{G \supseteq F} \mu(F,G) h_G\right) \left(\sum_{G' \supseteq F'} \mu(F',G') h_{G'}\right) & F, F' \text{ incomparable} \\ \left(\sum_{G \supseteq F} \mu(F,G) h_G\right) h_{F'}^{\mathrm{rk}(F')-\mathrm{rk}(F)} & F \subsetneq F' \\ h_F^{\mathrm{rk}(F)} & F \in \mathcal{L}_M \backslash \{\emptyset\}. \end{cases}$$

Proof. Let $S_{\mathrm{FY}} := \mathbb{R}\left[z_F : F \in \mathcal{L}_M \backslash \{\emptyset\}\right]$ and $S_\nabla := \mathbb{R}\left[h_F : F \in \mathcal{L}_M \backslash \{\emptyset\}\right]$, and define $\varphi : S_{\mathrm{FY}} \to S_\nabla$ to be the substitution $z_F \mapsto -\sum_{G \supseteq F} \mu(F,G) h_G$. Observe that φ is lower triangular with -1's on the diagonal when the variables z_F and h_F are written in descending order with respect to $>$. Hence, if $f \in S$ with initial monomial $z_{F_1}^{e_1} \cdots z_{F_k}^{e_k}$, then the initial monomial of $\varphi(f)$ is $h_{F_1}^{e_1} \cdots h_{F_k}^{e_k}$. The proposition follows from the fact that the elements of the Gröbner basis above are the images under φ of the elements of the Gröbner basis given in Theorem 3.9. □

As a result, we obtain an $\mathbb{R}$-basis of $A^\bullet_\nabla(M)$ consisting of monomials that are not initial in the Gröbner basis.

Corollary 3.11. *For $c \in \mathbb{Z}_{\leq 0}$, a monomial $\mathbb{R}$-basis for the degree c part $A^c_\nabla(M)$ of the Chow ring $A^\bullet_\nabla(M)$ of a matroid M is*

$$\left\{h_{F_1}^{a_1} \cdots h_{F_k}^{a_k} \,\middle|\, \sum a_i = c,\, \emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k,\, 1 \leq a_i < \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1})\right\}.$$

We call this basis of $A^\bullet_\nabla(M)$ the *nested basis* of the Chow ring of M.

3.3 Combinatorial Interpretations of the Chow Ring

3.3.1 Matroid quotients, matroid intersections and simplicial generators

Definition 3.6. Let M and M' be matroids on a common ground set E. M' is a *(matroid) quotient* of M, written $f : M \twoheadrightarrow M'$, if any of the following equivalent conditions hold:

1. There is a matroid N on $E' \supseteq E$ such that $M = N\backslash(E'\backslash E)$ and $M' = N/(E'\backslash E)$,
2. every flat of M' is also a flat of M,
3. the Bergman fan $\Sigma_{M'}$ is a subfan of Σ_M,
4. $\mathrm{rk}_{M'}(B) - \mathrm{rk}_{M'}(A) \leq \mathrm{rk}_M(B) - \mathrm{rk}_M(A)$ for every $A \subseteq B \subseteq E$,
5. every circuit of M is a union of circuits of M',
6. for all $A \subseteq E$, $\mathrm{cl}_M(X) \subseteq \mathrm{cl}_{M'}(X)$.

Remark that M' is a quotient of M iff M and M' are on the same ground set E and the identity map on E is a strong map $M \twoheadrightarrow M'$. Any matroid M on E is a quotient of the Boolean matroid $U_{|E|,E}$ as the Bergman fan of Σ_M is a subfan of $\Sigma_{A_n} = \Sigma_{U_{|E|,E}}$.

By definition, a matroid quotient $f : M \twoheadrightarrow M'$ defines an inclusion of fans $\iota_f : \Sigma_{M'} \hookrightarrow \Sigma_M$, defining an injective pushforward $\iota_{f*} : \mathrm{MW}_\bullet(\Sigma_{M'}) \hookrightarrow \mathrm{MW}_\bullet(\Sigma_M)$, which is a $A^\bullet(M)$-module map via the pullback map $\iota_f^* : A^\bullet(M) \twoheadrightarrow A^\bullet(M')$. For $\Delta \in \mathrm{MW}_\bullet(\Sigma_{M'})$, we write $\Delta(M') := \Delta \in \mathrm{MW}_\bullet(\Sigma_{M'})$ and $\Delta(M) := \iota_{f*}\Delta \in \mathrm{MW}_\bullet(\Sigma_M)$ when we need to distinguish between considering it as an element of $\mathrm{MW}_\bullet(\Sigma_{M'})$ and $\mathrm{MW}_\bullet(\Sigma_M)$. For example, that the pushforward map ι_{f*} is a $A^\bullet(M)$-module map reads $x_F(M') \cap \Delta(M') =$

$x_F(M) \cap \Delta(M)$ for $F \in \mathcal{L}_M$.

Definition 3.7. For a matroid quotient $f : M \twoheadrightarrow M'$, the *$f$-nullity* of $A \subseteq E$ is

$$n_f(A) := \mathrm{rk}_M(A) - \mathrm{rk}_{M'}(A).$$

We say that M' is an *elementary (matroid) quotient* of M if $n_f(E) = 1$, or equivalently if $\mathrm{rk}(M') = \mathrm{rk}(M) - 1$.

A remarkable property of matroid quotients is given in [Hig68], which states that any matroid quotient can be factorized into a sequence of elementary quotients.

Proposition 3.12. *Any matroid quotient $f : M \twoheadrightarrow M'$ has a canonical factorization into elementary quotients*

$$M = M_c \twoheadrightarrow \cdots \twoheadrightarrow M_1 \twoheadrightarrow M_0 = M'$$

called the Higgs factorization *of f, whose constituent matroids M_i can be described in two equivalent ways:*

1. *by the bases:* $\mathcal{B}(M_i) = \{A \subseteq E \,|\, A \in \mathcal{S}(M') \cap \mathcal{I}(M),\ |A| = \mathrm{rk}(M') + i\}$,
2. *inductively by the flats:* $\mathcal{L}_{M_{i+1}} = \mathcal{L}_{M_i} \cup \{F \in \mathcal{L}_M \,|\, \mathrm{rk}_M(F) = \mathrm{rk}_{M_i}(F)\}$.

The elementary quotients are cryptomorphic to modular cuts.

Definition 3.8. A *modular cut* of a matroid M is a nonempty collection of flats $\mathcal{K} \subseteq \mathcal{L}_M$ satisfying

1. if $F_1 \in \mathcal{K}$ and $F_2 \supseteq F_1$, then $F_2 \in \mathcal{K}$, and
2. if $F_1, F_2 \in \mathcal{K}$ and $\mathrm{rk}_M(F_1) + \mathrm{rk}_M(F_2) = \mathrm{rk}_M(F_1 \vee F_2) + \mathrm{rk}_M(F_1 \wedge F_2)$, then $F_1 \wedge F_2 \in \mathcal{K}$, where $\vee$ and $\wedge$ are the join and meet in $\mathcal{L}_M$, respectively.

Proposition 3.13. *The elementary quotients of M correspond bijectively to the modular cuts of M as follows: Given a matroid M and a modular cut $\mathcal{K}$ of M, we obtain an elementary quotient $M \twoheadrightarrow M'$ by taking*

$$\mathcal{L}_{M'} := \{F \in \mathcal{L}_M : F \text{ is not covered by an element of } \mathcal{K}\} \cup \mathcal{K}.$$

Conversely, given an elementary quotient $f : M \twoheadrightarrow M'$, one recovers the modular cut $\mathcal{K}$ of M defining the quotient $M \twoheadrightarrow M'$ by

$$\mathcal{K} = \{F \in \mathcal{L}_{M'} : n_f(F) = 1\}.$$

We denote by $M \overset{\mathcal{K}}{\twoheadrightarrow} M'$ the elementary quotient of M given by a modular cut $\mathcal{K}$ of M. The following characterizations of Higgs factorization in terms of modular cuts are given in [KK78].

Proposition 3.14. *Let $f : M \twoheadrightarrow M'$ be a matroid quotient on E with $n_f(E) = c$. The following are equivalent for a factorization $M = M_c \overset{\mathcal{K}_c}{\twoheadrightarrow} \cdots \overset{\mathcal{K}_2}{\twoheadrightarrow} M_1 \overset{\mathcal{K}_1}{\twoheadrightarrow} M_0 = M'$ of f into elementary quotients:*

1. *$M = M_c \overset{\mathcal{K}_c}{\twoheadrightarrow} \cdots \overset{\mathcal{K}_2}{\twoheadrightarrow} M_1 \overset{\mathcal{K}_1}{\twoheadrightarrow} M_0 = M'$ is the Higgs factorization of f,*
2. *$\mathcal{K}'_c \subseteq \cdots \subseteq \mathcal{K}'_2 \subseteq \mathcal{K}'_1$ where $\mathcal{K}'_i = \{A \subseteq E : \mathrm{cl}_{M_i}(A) \in \mathcal{K}_i\}$ and $\mathcal{K}_i = \mathcal{K}'_i \cap \mathcal{L}_{M_i}$,*
3. *$\mathcal{K}_i = \{F \in \mathcal{L}_{M_i} : n_f(F) \geq i\}$ for $i = 1, \cdots, c$.*

Note that for any $F \in \mathcal{L}_M$, the interval $[F, E] \subseteq \mathcal{L}_M$ forms a modular cut of M, and we call the resulting elementary quotient, denoted $T_F(M)$, the *principal truncation* of M associated to F.

Proposition 3.15. *The principal truncation $T_F(M)$ of a matroid M associated to $F \in \mathcal{L}_M$ has bases*

$$\mathcal{B}(T_F(M)) = \{B \backslash f : B \in \mathcal{B}(M),\, f \in B \cap F \neq \emptyset\}.$$

Definition 3.9. For two matroids M, N on a common ground set E, the *matroid intersection* of M and N is a matroid $M \wedge N$ on E whose spanning sets are

$$\mathcal{S}(M \wedge N) = \{S \cap S' \,|\, S \in \mathcal{S}(M),\, S' \in \mathcal{S}(N)\}.$$

Matroid intersection behaves well in relation to Minkowski weights. Recall that two loopless matroids M, N define Minkowski weights $\Delta_M, \Delta_N \in \mathrm{MW}^\bullet(\Sigma_{A_n})$. By Theorem 3.5, one can consider the product $\Delta_M \cdot \Delta_N$ under the identification $\mathrm{MW}^\bullet(\Sigma_{A_n}) \cong A^\bullet(\Sigma_{A_n})$.

Proposition 3.16 ([Spe08]). *For loopless matroids M, N on a common ground set E, we have*

$$\Delta_M \cdot \Delta_N = \begin{cases} \Delta_{M \wedge N} & \text{if } M \wedge N \text{ is loopless,} \\ 0 & \text{otherwise.} \end{cases}$$

For $S \subseteq E$, we denote by H_S the matroid with bases

$$\mathcal{B}(H_S) = \{E \backslash i : i \in S\},$$

or equivalently, $H_S = U_{|E\setminus S|,E\setminus S} \oplus U_{|S|-1,S}$. By comparing with the bases given in Proposition 3.15, we have the following connection between principal truncations and matroid intersections.

Proposition 3.17. *Let M be a loopless matroid on E, and let $S \subseteq E$ and $F = \mathrm{cl}_M(S)$ the closure of S in M. Then*

$$T_F(M) = M \wedge H_S$$

A nef divisor $D \in A^1(\Sigma)$ on a complete fan Σ corresponds via Theorem 3.5 to a Minkowski weight $\delta(D) \in \mathrm{MW}^1(\Sigma)$ of codimension one. The following description of $\delta(D)$ can be found in Proposition 3.3.2 & Theorem 6.7.7 in [MS15].

Theorem 3.18. *Let Δ_Σ be the fundamental class of the complete fan Σ, and D a nef divisor on Σ such that P_D is a lattice polytope. Then we have $\delta(D) = D \cap \Delta_\Sigma = \Delta_{P_D}$ where for each $\tau \in \Sigma(n-1)$, we have*

$$\Delta_{P_D}(\tau) = \begin{cases} l(P_D(\sigma)) & \text{if there is } \sigma \in \Sigma_{P_D}(n-1) \text{ such that } |\tau| \subseteq |\sigma|, \\ 0 & \text{otherwise,} \end{cases}$$

where $l(P_D(\sigma))$ denotes the lattice length of the edge of P_D corresponding to σ, i.e. the number of lattice points minus 1.

The following two lemmas convey the geometry origin of the presentation $A^\bullet_\nabla(M)$: Multiplying with h_F corresponds to the principal truncation by F, which is, when realizable and realized as hyperplane arrangements, intersecting by a general hyperplane containing the subspace corresponding to F. The notation "h" stands therefore for "hyperplane".

Lemma 3.19. *For $\emptyset \neq S \subseteq E$, consider $h_S \in A^\bullet_\nabla(\Sigma_{A_n})$. We have*

$$h_S \cap \Delta_{U_{|E|,E}} = \Delta_{H_S}.$$

Proof. Observe that the translating of the polytope ∇_S of the nef divisor h_S by $\mathbf{1}$ gives

$$\nabla_S + \mathbf{1} = \mathrm{conv}(\mathbf{e}_{E\setminus i} : i \in S) \subset \mathbb{R}^E,$$

which is the matroid base polytope $\mathcal{P}(H_S)$ of the matroid H_S. That is, we have $h_S = D_{\mathcal{P}(H_S)}$ as in Theorem 3.3. As the edges of matroid base polytopes are parallel translates of $\mathbf{e}_i - \mathbf{e}_j$ by Theorem 2.7, and hence their lattice length are all equal to 1. Thus, Theorem 3.18 implies that $h_S \cap \Delta_{U_{|E|,E}} =$

$\Delta_{\mathcal{P}(H_S)} \in \mathrm{MW}^1(\Sigma_{A_n})$ where $\Delta_{\mathcal{P}(H_S)} = 1$ if σ is contained in some cone of same dimension in the normal fan $\Sigma_{\mathcal{P}(H_S)}$ and 0 otherwise.
Because the Bergman class Δ_{H_S} has the weight 1 or 0 as well, what is left to show is that Σ_{H_S} and the codimension 1 skeleton $\Sigma_{\mathcal{P}(H_S)}(n-1)$ of the normal fan $\Sigma_{\mathcal{P}(H_S)}$ have the same support. We have $|\Sigma_{U_{n-1,n}}| = |\Sigma_{\mathcal{P}(U_{n-1,n})}(n-1)| \subset \mathbb{R}^n/\mathbb{R}\mathbf{1}$, because $\mathcal{P}(U_{n-1,n})$ is an $(n-1)$-dimensional simplex whose edge $\operatorname{conv}\{\mathbf{1}-\mathbf{e}_i, \mathbf{1}-\mathbf{e}_j\}$, $i \neq j \in [n]$ has outer normal cone as union of $(n-1)$-dimensional cones of $\Sigma_{U_{n-1,n}}$ corresponding to the flags with maximal element $[n]\backslash\{i,j\}$. Then our desired equality $|\Sigma_{H_S}| = |\Sigma_{\mathcal{P}(H_S)}(n-1)|$ follows from observing that $H_S = U_{|E\backslash S|, E\backslash S} \oplus U_{|S|-1,S}$. Remark that $\mathcal{L}_{M\oplus M'} \cong \mathcal{L}_M \times \mathcal{L}_{M'}$ and $\mathcal{P}(M \oplus M') = \mathcal{P}(M) \times \mathcal{P}(M')$. □

Lemma 3.20. *Let $h_S \in A^\bullet_\nabla(U_{|E|,E})$ for $\emptyset \neq S \subseteq E$, and let M be a loopless matroid on E. Let $F = \mathrm{cl}_M(S)$ be the closure of S in M. We have*

$$h_S \cap \Delta_M = \Delta_{T_F(M)} \quad \textit{and} \quad h_F(M) \cap \Delta_M(M) = \Delta_{T_F(M)}(M).$$

Proof. We have

$$\begin{aligned} h_S \cap \Delta_M &= h_S \cap \left(x \cap \Delta_{U_{|E|,E}}\right) = (h_S \cdot x) \cap \Delta_{U_{|E|,E}} \\ &= \left(h_S \cap \Delta_{U_{|E|,E}}\right) \cdot \left(x \cap \Delta_{U_{|E|,E}}\right) = \left(h_S \cap \Delta_{U_{|E|,E}}\right) \cdot \Delta_M \\ &= \Delta_{H_S} \cdot \Delta_M = \Delta_{H_S \wedge M} = \Delta_{T_F(M)}. \end{aligned}$$

The first, the third and the fourth equalities follow from the Poincaré duality isomorphism δ in Theorem 3.5 where we set $x := \delta^{-1}(\Delta_M)$. The second equality follows from the Kronecker duality in Theorem 3.2 and the definition of the cap product. And the fifth, sixth and seventh equalities follow from Lemma 3.20, Proposition 3.16 and Proposition 3.17, respectively.
The second equation is a consequence of ι_{M*} being a $A^\bullet(\Sigma_{A_n})$-module map via the pullback map $\iota_M^* : A^\bullet(\Sigma_{A_n}) \twoheadrightarrow A^\bullet(M)$. □

3.3.2 Nested basis and relative nested quotients

Recall that from Corollary 3.11,

$$\left\{ h_{F_1}^{a_1} \cdots h_{F_k}^{a_k} \,\middle|\, \sum a_i = c,\, \emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k,\, 1 \leq a_i < \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) \right\}$$

is a monomial $\mathbb{R}$-basis for the degree c part $A^c_\nabla(M)$ of $A^\bullet_\nabla(M)$. This basis of $A^\bullet_\nabla(M)$ is called the nested basis of the Chow ring of M. We will show that these monomials in the basis allow for a combinatorial interpretation as a distinguished set of matroid quotients of M.

Definition 3.10. Let $f : M \twoheadrightarrow M'$ be a matroid quotient on a ground set E. An *f-cyclic flat* of f is a flat $F \in \mathcal{L}_{M'}$ such that F is minimal (with respect to inclusion) among the flats $F' \in \mathcal{L}_{M'}$ such that $n_f(F') = n_f(F)$. The set of f-cyclic flats is denoted by $\mathrm{cyc}(f)$. A matroid M' is a *relative nested quotient* of M if the f-cyclic flats of M' form a chain.

Proposition 3.21. *The data of the f-cyclic flats, their f-nullities and the matroid M determine the quotient $f : M \twoheadrightarrow M'$.*

Proof. Let $c = n_f(E)$. We show that from the data given, one can recover the Higgs factorization $M = M_c \xrightarrow{\mathcal{K}_c}\!\!\!\!\twoheadrightarrow \cdots \xrightarrow{\mathcal{K}_2}\!\!\!\!\twoheadrightarrow M_1 \xrightarrow{\mathcal{K}_1}\!\!\!\!\twoheadrightarrow M_0 = M'$ of f. By Proposition 3.14, the modular cuts $\mathcal{K}_i$ are given as follows for all $i = 1, \cdots, c$:

$$\begin{aligned}\mathcal{K}_i &= \{G \in \mathcal{L}_{M_i} : n_f(G) \geq i\} \\ &= \{G \in \mathcal{L}_{M_i} : G \supseteq F \text{ for some } F \in \mathrm{cyc}(f) \text{ with } n_f(F) \geq i\}.\end{aligned}$$

In other words, we have recovered the modular cuts defining the Higgs factorization of $M \twoheadrightarrow M'$ from the given data. □

We now show that the nested basis of $A^{\bullet}_{\nabla}(M)$ is in bijection with the set of relative nested quotients of M, moreover, the bijection respects linear independence.

Theorem 3.22. *Let M be a loopless matroid of rank $r = d + 1$. For each $0 \leq c \leq d$, the cap product map*

$$A^{c}_{\nabla}(M) \to \mathrm{MW}_{d-c}(\Sigma_M), \;\; \xi \mapsto \xi \cap \Delta_M$$

induces a bijection between the monomial basis for $A^{c}_{\nabla}(M)$ given in Corollary 3.11 and the set of Bergman classes $\Delta_{M'}$ of loopless relative nested quotients $M' \twoheadleftarrow M$ with $\mathrm{rk}(M') = \mathrm{rk}(M) - c$.

Proof. Let $h^{a_1}_{F_1} \cdots h^{a_k}_{F_k}$ be an element of the monomial basis given in Corollary 3.11. By Lemma 3.20, Proposition 3.17 and Proposition 3.16, we have

$$h^{a_1}_{F_1} \cdots h^{a_k}_{F_k} \cap \Delta_M = \Delta^{a_1}_{T_{F_1}(M)} \cdots \Delta^{a_k}_{T_{F_k}(M)} = \Delta^{a_1}_{M \wedge H_{F_1}} \cdots \Delta^{a_k}_{M \wedge H_{F_1}} = \Delta_{M'}(M),$$

where

$$M' = M \wedge^{a_k} H_{F_k} \cdots \wedge^{a_1} H_{F_1} := M \wedge \underbrace{H_{F_k} \wedge \cdots \wedge H_{F_k}}_{a_k \text{ times}} \wedge \cdots \wedge \underbrace{H_{F_1} \wedge \cdots \wedge H_{F_1}}_{a_1 \text{ times}}.$$

By Proposition 3.17, $f : M' \twoheadleftarrow M$ is a matroid quotient with Higgs factorization

$$M \underbrace{\xrightarrow{[F_k,E]}\!\!\!\!\twoheadrightarrow \cdots \xrightarrow{[F_k,E]}\!\!\!\!\twoheadrightarrow}_{a_k \text{ times}} \cdots \underbrace{\xrightarrow{[F_1,E]}\!\!\!\!\twoheadrightarrow \cdots \xrightarrow{[F_1,E]}\!\!\!\!\twoheadrightarrow}_{a_1 \text{ times}} M'.$$

The description of modular cuts in Proposition 3.13 implies that $\mathrm{cyc}(f) = \{F_1, \ldots, F_k\}$ and $n_f(F_j) = \sum_{i=1}^{j} a_i$. The inequalities $1 \le a_i < \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1})$ ensure that F_i is a flat in $M \wedge^{a_k} H_{F_k} \cdots \wedge^{a_{i+1}} H_{F_{i+1}}$, and in particular $\mathrm{rk}(F_1) - a_1 > 0$ ensures loopless.
Conversely, from the construction given in the proof of Proposition 3.21, one sees that if $f : M' \twoheadleftarrow M$ is a loopless nested matroid quotient with $\mathrm{cyc}(f) = \{F_1 \subsetneq \cdots \subsetneq F_k\}$, then $M' = M \wedge^{a_k} H_{F_k} \cdots \wedge^{a_1} H_{F_1}$ where $a_j = n_f(F_j) - n_f(F_{j-1})$ for $j > 1$, and $a_1 = n_f(F_1)$. □

Proposition 3.23. *The elements*

$$\{\Delta_{M'} : M' \text{ is a loopless relative nested quotient of } M\}$$

are linearly independent in $\mathrm{MW}_\bullet(\Sigma_{A_n})$.

Proof. For a loopless relative nested matroid quotient $f : M' \twoheadleftarrow M$ of corank $c = \mathrm{rk}(M) - \mathrm{rk}(M')$ given by f-cyclic flats $\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k$ with ranks $r_i := \mathrm{rk}_{M'}(F_i)$, $i = 0, \ldots, k$, define

$$\gamma(f) := (d_i)_{i=1,\ldots,r}, \quad d_i := \begin{cases} r_i - r_{i-1} & \text{if } i \le k, \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $\mathfrak{M}_r$ the set of loopless relative nested matroid quotients $f : M' \twoheadleftarrow M$ of rank $\mathrm{rk}(M') = r$.
Assume that we have a linear relation

$$\sum_{f : M \twoheadrightarrow M' \in \mathfrak{M}_r} a_{M'} \Delta_{M'} = 0,$$

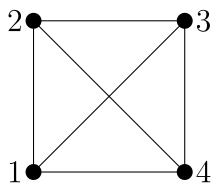
we show by lexicographic induction on $\gamma(f)$ that $a'_M = 0$ for any $M' \in \mathfrak{M}_r$.
For the base case, consider the case of $f : M' \twoheadleftarrow M$ with f-cyclic flats $\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k$ satisfying

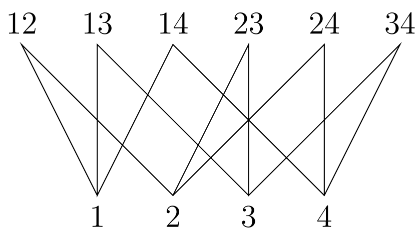
$$\gamma(f) = (\underbrace{1, \ldots, 1}_{k \text{ many}}, 0, \ldots, 0).$$

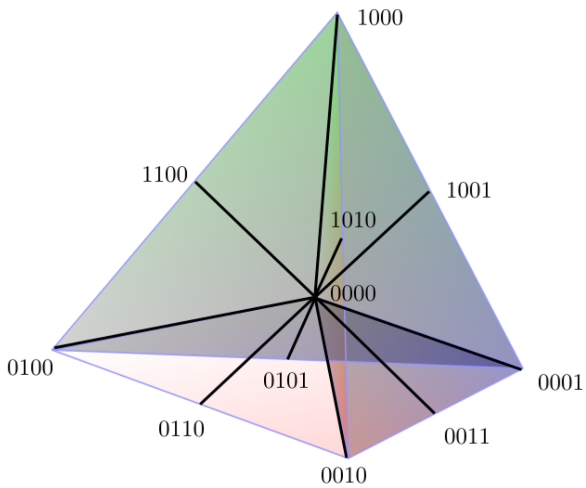
Extend the chain of f-cyclic flats of M' to any maximal chain of flats in M', and consider a loopless relative nested matroid quotient $g : N' \twoheadleftarrow M$

also containing this chain as a maximal chain of flats. We show that $N' = M'$. Note that by construction $\mathrm{rk}_{N'}(F_i) = \mathrm{rk}_{M'}(F_i)$ for all $0 \leq i \leq k$. By induction assume $F_0, \ldots, F_{j-1}$ are g-cyclic. If F_j is not g-cyclic, then it contains a g-cyclic flat G with the same g-nullity as that of F_j. But then $n_g(G) = n_g(F_j) = n_f(F_j) > n_f(F_{j-1}) = n_g(F_{j-1})$, implies $G \supsetneq F_{j-1}$, which contradicts $\mathrm{rk}_{N'}(G) < \mathrm{rk}_{N'}(F_j) = \mathrm{rk}_{N'}(F_{j-1}) + 1$. Thus, all F_i's are g-cyclic as well with $\mathrm{rk}_{N'}(F_i) = i$, and there are no other g-cyclic flats since $n_f(E) = n_g(E)$. This means that M' is the unique element in $\mathfrak{M}_r$ such that the $(r-1)$-dimensional cone corresponding to the maximal flag is in the support of $\Delta_{M'}$, which implies $a_{M'} = 0$.
Now suppose $\gamma(f) = (d_1, \ldots, d_r) > (0, \ldots, 0)$ and consider $g : N' \twoheadleftarrow M$ that has a maximal chain of flats that is also a maximal chain in $\mathcal{L}_{M'}$ containing the f-cyclic flats. We show that $N' \neq M'$ then $\gamma(g) <_{lex} \gamma(f)$, thereby completing the induction to conclude that $a_{M'} = 0$ for any $M' \in \mathfrak{M}_r$.
Let $\gamma(g) = (c_1, \ldots, c_r)$, and suppose $1 \leq j \leq k$ is the minimum j such that F_j is not g-cyclic, which exists since $N' \neq M'$. By the same arguments given in the case of $\gamma(f) = (1, \ldots, 1, 0, \ldots, 0)$, we then have a g-cyclic flat G such that $F_{j-1} \subsetneq G \subsetneq F_j$, which decreases c_j by at least one. Hence, $\gamma(g) < \gamma(f)$, as desired. □

Example 3.24 ($U_{3,4}$)**.** Let $M = U_{3,4}$ be the rank 3 uniform matroid on $E = \{1, 2, 3, 4\}$. The following are the geometric representation as the affine plane $\mathbb{F}_2^2$, the lattice of nonempty proper flats and the Bergman fan in $\mathbb{R}^4/\mathbb{R}\mathbf{1}$.







The Chow ring $A^\bullet(M) = A^0(M) \oplus A^1(M) \oplus A^2(M)$ is the polynomial ring $\mathbb{R}[x_1, x_2, x_3, x_4, x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}]$ modulo the ideal generated by the squarefree quadratic monomials

$$x_1x_2, x_1x_3, \ldots, x_3x_4, \quad x_{12}x_{13}, x_{12}x_{14}, \ldots, x_{24}x_{34}, \quad x_1x_{23}, x_1x_{24}, x_1x_{34}, \ldots, x_4x_{23}$$

and the linear forms

$$x_1+x_{13}+x_{14}-x_2-x_{23}-x_{24},\ x_1+x_{12}+x_{14}-x_3-x_{23}-x_{24},\ x_1+x_{12}+x_{13}-x_4-x_{24}-x_{34}.$$

As the three linear forms are linearly independent, we have $\dim A^1(M) = 10-3 = 7$ and $\dim A^0(M) = \dim A^2(M) = 1$.
The Minkowski weights satisfy $\mathrm{MW}_0(M) \cong \mathrm{MW}_2(M) \cong \mathbb{R}$, and $\mathrm{MW}_1(M)$ is the solution space of

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \mathbf{x} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

where $\lambda \in \mathbb{R}$. Therefore, $\dim \mathrm{MW}_1(M) = 7$, as desired. The cap product $x_F \cap \Delta_M \in \mathrm{MW}_1(M)$ maps a flat $F' \in \mathcal{L}_M \backslash \{\emptyset, E\}$ to 1 if $F \neq F'$ comparable, to 0 if F, F' are incomparable, to -1 if $F = F'$ is of rank 2, and to -2 if $F = F'$ is of rank 1.
The nested basis of $A^1_\nabla(M)$ is $\{h_{12}, h_{13}, h_{14}, h_{23}, h_{24}, h_{34}, h_E\}$. One computes that $h_{12} = -z_{12} - z_E = x_1 + x_{13} + x_{14}$ and

$$h_{12} \cap \Delta_M(F) = 1 \text{ if } F \in \{12, 3, 4\} \text{ and } 0 \text{ otherwise.}$$

This is exactly the Bergman class of the relative nested quotient $T_{12}(M)$ of M given by the modular cut $\{12, E\}$. And the Minkowski weight $h_E \cap \Delta_M$ maps a nonempty proper flat F to 1 if $F \in \{1, 2, 3, 4\}$ and 0 otherwise, it is the Bergman class of the truncation $T_E(M)$. Moreover, one sees that $h_{12} \cap \Delta_M, \ldots, h_E \cap \Delta_M$ are linearly independent as every flat $F \in \mathcal{L}_M$ of rank 2 is only in the support of $h_F \cap \Delta(M)$.

Chapter 4

Hodge Theory of Matroids

4.1 Poincaré Duality

Let X be a compact orientable connected manifold of dimension d, and let $H_i(X)$ and $H^i(X)$ be the i-th singular homology and cohomology space with coefficients in $\mathbb{R}$, respectively. The Poincaré duality theorem states that a choice of a fundamental class $[X] \in H^d(X)$ such that $H^d(X) \cong \mathbb{R}\{[X]\}$ induces isomorphisms $H^k(X) \xrightarrow{\sim} H_{d-k}(X)$, $\varphi \mapsto \varphi \cap [X]$. Equivalently, there are non-degenerate pairings $H^k(X) \times H^{d-k}(X) \to H_0(X) = \mathbb{R}$, $(\varphi, \psi) \mapsto (\phi \cup \psi) \cap [X]$, where $\cup$ and $\cap$ are the cup and cap product, respectively. This property is the first component of the Kähler package. In this section we will show that it is satisfied by the Chow ring of a matroid as well. We refer to [Hat02] for the background in algebraic topology and [MS+05] for an abstract theory.

Definition 4.1. A graded finite (commutative) $\mathbb{K}$-algebra $A^\bullet = \bigoplus_{i=0}^d A^i$ is a *(graded) Poincaré duality algebra* of dimension d if

(1) $A^0 = \mathbb{K}$, and

(2) there exists a $\mathbb{K}$-linear isomorphism $\int : A^d \xrightarrow{\sim} \mathbb{K}$, called the *degree map* of $A^\bullet$, such that

$$A^i \times A^{d-i} \to A^d \xrightarrow{\sim} \mathbb{K}, \quad (a, b) \mapsto \int ab$$

is a non-degenerate pairing for all $0 \leq i \leq d$.

The second condition is equivalent to

(2') the map $A^k \to \mathrm{Hom}(A^{d-k}, \mathbb{K})$, $\xi \mapsto (\zeta \mapsto \int \xi \cdot \zeta)$ is a $\mathbb{K}$-isomorphism for all $0 \leq k \leq d$.

By Theorem 3.5, the Chow ring $A^\bullet(\Sigma)$ of a smooth complete rational fan Σ is a Poincaré duality algebra with degree map $\int_\Sigma$.

Definition 4.2. Given a commutative ring R and $f \in R$, the *f-transport* of R, denoted by $\langle f \rangle$ by abuse of notation, is a ring whose elements are those of the principal ideal $\langle f \rangle \subset R$ with multiplication defined by $af \cdot bf = (ab)f$. That is,

$$\langle f \rangle \cong R/\mathrm{Ann}_R(f), \quad \text{where } \mathrm{Ann}_R(f) := \{r \in R \,|\, rf = 0\}.$$

The ring $\langle f \rangle$ comes with a natural surjection $R \twoheadrightarrow \langle f \rangle$ of rings and an injection $\langle f \rangle \hookrightarrow R$ of R-modules.

Proposition 4.1. *If $(A^\bullet, \int)$ is a Poincaré duality algebra of dimension d, and $f \in A^\bullet$ is a homogeneous element of degree k, then the f-transport $\langle f \rangle$ of $A^\bullet$ is a Poincaré duality algebra of dimension $d - k$ with the* induced degree map $\int_f$ *defined as $\int_f (a + \mathrm{Ann}(f)) := \int af$ for $a \in A^{d-k}$.*

Proof. If $\int_f (af \cdot bf) = \int_f (ab)f = \int abf = 0$ for all $a \in A^i$, equivalently, for all $af \in \langle f \rangle^i$, then $bf = 0$ by the Poincaré duality property of $(A^\bullet, \int)$. □

Let M be a loopless matroid of rank $r = d + 1$ on a ground set $E = \{0, \ldots, n\}$. We show that the Chow ring $A^\bullet(M)$ is the Δ_M-transport of $A^\bullet(X_n) \cong \mathrm{MW}^\bullet(\Sigma_{A_n})$ and conclude that it is a Poincaré duality algebra. This is a consequence of Theorem 3.22: first show that the cap product map $A^\bullet_\nabla(M) \xrightarrow{\cdot \cap \Delta_M} \mathrm{MW}_{d-\bullet}(\Sigma_M) \xhookrightarrow{\iota_{M*}} \mathrm{MW}_{d-\bullet}(\Sigma_{A_n})$ factors as $A^\bullet_\nabla(M) \twoheadrightarrow \langle \Delta_M \rangle \hookrightarrow \mathrm{MW}_{d-\bullet}(\Sigma_{A_n})$, and the injectivity follows from Proposition 3.23.

Proposition 4.2. *We have the following commutative diagram of graded algebras:*

$$\begin{array}{ccccc}
A^\bullet(\Sigma_{A_n}) & \xrightarrow{\sim} & \mathrm{MW}^\bullet(\Sigma_{A_n}) & & \\
\downarrow{\scriptstyle \iota_M^*} & & \downarrow & & \\
A^\bullet(M) & \twoheadrightarrow & \langle \Delta_M \rangle & \hookrightarrow & \mathrm{MW}_{d-\bullet}(\Sigma_{A_n})
\end{array}
\qquad
\begin{array}{ccc}
x_S & \longmapsto & x_S \cap \Delta_{\Sigma_{A_n}} \\
\downarrow & & \downarrow \\
x_S(M) & \longmapsto & x_S \cap \Delta_M
\end{array}$$

Proof. The right vertical map is the canonical surjective map from the ring $\mathrm{MW}^\bullet(\Sigma_{A_n})$ to the Δ_M-transport $\langle \Delta_M \rangle$, which also canonically injects into

$\mathrm{MW}^{\bullet-(n-d)}(\Sigma_{A_n}) = \mathrm{MW}_{d-\bullet}(\Sigma_{A_n})$. The left vertical map ι_M^* is the pullback map, and the top horizontal isomorphism is the cap product with $\Delta_{\Sigma_{A_n}}$, as in Theorem 3.5. We now construct the bottom horizontal map as the canonical ring map induced from the fact that the kernel of $A^\bullet(\Sigma_{A_n}) \twoheadrightarrow \langle \Delta_M \rangle$ contains the kernel of $\iota_M^* : A^\bullet(\Sigma_{A_n}) \twoheadrightarrow A^\bullet(M)$, which is exactly the quotient map

$$\iota_M^* : A^\bullet(\Sigma_{A_n}) \twoheadrightarrow A^\bullet(\Sigma_{A_n}) / \langle x_S : \emptyset \subsetneq S \subsetneq E \text{ is not a flat of } M \rangle.$$

What is left to compute is $x_S \cap \Delta_M = 0$ if S is not a flat of M. For a $(d-1)$-dimensional cone $\sigma_{\mathcal{F}} \in \Sigma_M(d-1)$ corresponding to a flag $\mathcal{F} = (\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_{d-1} \subsetneq E)$ of $d-1$ proper flats in M, the Minkowski weight $x_S \cap \Delta_M$ assigns the value $(t_{\Sigma_M} \Delta_M)(x_S x_{F_1} \cdots x_{F_{d-1}})$ to the cone $\sigma_{\mathcal{F}}$. As S is a not a flat of M, either S is incomparable to some F_i or S fits into a chain $\mathcal{F}_S := (F_1 \subsetneq \cdots \subsetneq F_i \subsetneq S \subsetneq F_{i+1} \subsetneq \cdots \subsetneq F_{d-1})$. In the first case the monomial $x_S x_{F_1} \cdots x_{F_{d-1}}$ is zero in $A^\bullet(\Sigma_{A_n})$, and in the second case we have $\Delta_M(\sigma_{\mathcal{F}_S}) = 0$ as $\mathcal{F}_S$ is not a chain of flats in M.
Thus, we have $\ker(A^\bullet(\Sigma_{A_n}) \twoheadrightarrow A^\bullet(M)) \subseteq \ker(A^\bullet(\Sigma_{A_n}) \twoheadrightarrow \langle \Delta_M \rangle)$, so that we have the induced canonical surjective map $A^\bullet(M) \twoheadrightarrow \langle \Delta_M \rangle$ defined by $x_F(M) \mapsto x_F \cap \Delta_M$. □

Corollary 4.3. *In the simplicial presentation, the diagram of Proposition 4.2 is*

$$\begin{array}{ccccc} A^\bullet_\nabla(\Sigma_{A_n}) & \xrightarrow{\sim} & \mathrm{MW}^\bullet(\Sigma_{A_n}) & & \\ \downarrow{\scriptstyle \iota_M^*} & & \downarrow & & \\ A^\bullet_\nabla(M) & \twoheadrightarrow & \langle \Delta_M \rangle & \hookrightarrow & \mathrm{MW}_{d-\bullet}(\Sigma_{A_n}) \end{array} \qquad \begin{array}{ccc} h_S & \longmapsto & \Delta_{H_S} \\ \downarrow & & \downarrow \\ h_{\mathrm{cl}_M(S)} & \longmapsto & \Delta_{T_{\mathrm{cl}_M(S)}(M)} \end{array}$$

Proof. It follows from Lemma 3.20. □

Theorem 4.4. *The Chow ring $A^\bullet(M)$ is the Δ_M-transport of* $\mathrm{MW}^\bullet(\Sigma_{A_n}) \cong A^\bullet(\Sigma_{A_n})$.

Proof. The surjective ring map $A^\bullet_\nabla(M) \twoheadrightarrow \langle \Delta_M \rangle$ is an isomorphism of $\mathbb{R}$-vector spaces as the bijection by cap product with Δ_M preserves linear independence, which is shown in Proposition 3.23. □

Corollary 4.5. *The Chow ring $A^\bullet(M)$ is a graded Poincaré duality algebra of dimension* $\mathrm{rk}(M) - 1$ *with $\int_M$ as the degree map.*

Proof. It follows from Theorem 3.5, Proposition 4.1 and Theorem 4.4. □

Corollary 4.6. *For each $0 \leq c \leq d$, the cap product map*

$$A^c(M) \to \mathrm{MW}_{d-c}(\Sigma_M), \quad \xi \mapsto \xi \cap \Sigma_M$$

is an isomorphism of $\mathbb{R}$-vector spaces. Thus, the Bergman classes of relative nested quotients form a basis of $\mathrm{MW}_\bullet(\Sigma_M)$.

Proof. By Corollary 4.5 and Theorem 3.2, we have the isomorphisms

$$A^c(M) \xrightarrow{\sim} \mathrm{Hom}(A^{d-c}(M), \mathbb{R}) \xrightarrow{\sim} \mathrm{MW}_{d-c}(\Sigma_M),$$

$$\xi \mapsto \left(\zeta \mapsto \int_M \zeta \cdot \xi = (\zeta \cdot \xi) \cap \Delta_M = \zeta \cap (\xi \cap \Delta_M)\right) \mapsto \xi \cap \Delta_M.$$

The second statement follows from Theorem 3.22. □

4.2 The Volume Polynomial

4.2.1 The volume polynomial of a matroid

Definition 4.3. Let $(A^\bullet, \int)$ be a graded Poincaré algebra of dimension d that is generated in degree 1, with a chosen presentation $A^\bullet = \mathbb{K}[x_1, \ldots, x_s]/I$ and a degree map $\int : A^d \to \mathbb{K}$. Then its *volume polynomial* VP_A is a multivariate polynomial in $\mathbb{K}[t_1, \ldots, t_s]$ defined by

$$\mathrm{VP}_A(t_1, \ldots, t_s) := \int (t_1 x_1 + \ldots + t_s x_s)^d,$$

where we extend the degree map $\int$ to $A[t_1, \ldots, t_s] \to \mathbb{K}[t_1, \ldots, t_s]$. As a function $A^1 \to \mathbb{K}$, this polynomial is the *volume function*

$$\mathrm{vol}_A : A^1 \to \mathbb{K}, \quad \ell \mapsto \int \ell^d.$$

We remark the following properties from [CLS11, §13.4]: If Σ is a smooth complete fan and D a nef divisor on Σ, then $\mathrm{vol}_{A^\bullet(\Sigma)}(D)$ is the normalized volume of the deformation P_D. If $(A^\bullet, \int)$ is a Poincaré duality algebra with a chosen presentation $A^\bullet = \mathbb{K}[x_1, \ldots, x_s]/I$, then the defining ideal I can be recovered from the volume polynomial VP_A by

$$I = \left\{ f(x_1, \ldots, x_s) \in \mathbb{K}[x_1, \ldots, x_s] \,\middle|\, f\left(\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_s}\right) \cdot \mathrm{VP}_A(t_1, \ldots, t_s) = 0 \right\}.$$

In this section, we deduce the formula for the volume polynomial VP_M^{∇} of $A^{\bullet}_{\nabla}(M)$. Let $E = \{0, \ldots, n\}$. A *transversal* of a family (repetitions allowed) $\{A_0, \ldots, A_m\}$ of subsets of E is a subset $I \subseteq E$ such that there exists a bijection $\phi : \{A_0, \ldots, A_m\} \to I$ satisfying $\phi(A_i) \in A_i$ for all $0 \leq i \leq m$. The following is a classic theorem of Rado which generalizes Hall's marriage theorem.

Theorem 4.7 (Rado's theorem)**.** *Let M be a matroid on E. A family of subsets $\{A_0, \ldots, A_m\}$ of E has a transversal $I \subseteq E$ that is independent in M iff*

$$\mathrm{rk}_M \left(\bigcup_{j \in J} A_j \right) \geq |J|, \quad \forall J \subseteq \{0, \ldots, m\}. \tag{4.1}$$

Proof. If the family $\{A_0, \ldots, A_m\}$ of E has an independent transversal $I \subseteq E$ with the bijection $\phi : \{A_0, \ldots, A_m\} \to I$ as in the definition, then for any subset $J \subseteq \{0, \ldots, m\}$,

$$\mathrm{rk}_M \left(\bigcup_{j \in J} A_j \right) \geq \mathrm{rk}_M \{\phi(A_j) : j \in J\} = |\{\phi(A_j) : j \in J\}| = |J|.$$

Conversely, assume that (4.1) holds. If all sets $A_j = \{a_j\}$ are singletons, then $\{a_j : j \in \{0, \ldots, m\}\}$ is the required transversal. Thus we may assume, without loss of generality, that $|A_0| \geq 2$. We will show that for some element x of A_0, the family $\{A_0 \backslash \{x\}, A_1, \ldots, A_m\}$ satisfies (4.1). By deleting the elements from each set in the family until remaining only singletons, the proof is finished.

Assume that no such element x exists. Then if x_1 and x_2 are distinct elements of A_0, there are nonempty subsets J_1 and J_2 of $\{1, \ldots, n\}$ such that

$$\mathrm{rk}_M \left((A_0 \backslash \{x_i\}) \cup \bigcup_{j \in J_i} A_j \right) < |J_i| + 1$$

for $i = 1, 2$. By submodularity,

$$\begin{aligned}
|J_1| + |J_2| &\geq \operatorname{rk}_M\left((A_0\backslash\{x_1\}) \cup \bigcup_{j\in J_1} A_j\right) + \operatorname{rk}_M\left((A_0\backslash\{x_2\}) \cup \bigcup_{j\in J_2} A_j\right) \\
&\geq \operatorname{rk}_M\left((A_0\backslash\{x_1\}) \cup \bigcup_{j\in J_1} A_j \cup (A_0\backslash\{x_2\}) \cup \bigcup_{j\in J_2} A_j\right) \\
&\quad + \operatorname{rk}_M\left(\left((A_0\backslash\{x_1\}) \cup \bigcup_{j\in J_1} A_j\right) \cap \left((A_0\backslash\{x_2\}) \cup \bigcup_{j\in J_2} A_j\right)\right) \\
&\geq \operatorname{rk}_M\left(A_0 \cup \bigcup_{j\in J_1\cup J_2} A_j\right) + \operatorname{rk}_M\left(\bigcup_{j\in J_1\cap J_2} A_j\right) \\
&\geq 1 + |J_1 \cup J_2| + |J_1 \cap J_2| = 1 + |J_1| + |J_2|,
\end{aligned}$$

contradiction. □

Proposition 4.8 (Dragon Hall-Rado condition). *Let M be a matroid on E, and $\{A_1, \ldots, A_m\}$ a family of subsets of E. There is an independent transversal $I \subseteq E\backslash\{e\}$ of $\{A_1, \ldots, A_m\}$ for every $e \in E$ iff*

$$\operatorname{rk}_M\left(\bigcup_{j\in J} A_j\right) \geq |J| + 1, \quad \forall \emptyset \subsetneq J \subseteq \{1, \ldots, m\}. \tag{4.2}$$

When the condition (4.2) is satisfied, we say that $\{A_1, \ldots, A_m\}$ satisfy the *dragon Hall-Rado condition* of M, or $\mathrm{DHR}(M)$ for short.

Proof. This follows from Rado's theorem and the observation that independent transversals $I \subseteq E\backslash\{e\}$ are the same as independent transversals of $\{A_1\backslash\{e\}, \ldots, A_m\backslash\{e\}\}$. □

Theorem 4.9. *Let $\{A_1, \ldots, A_d\}$ be a collection of subsets of E, and M a loopless matroid on E of rank $d+1$. Let $H_{A_1}, \ldots, H_{A_d}$ be matroids defined in Proposition 3.17. Then*

$$M \wedge H_{A_1} \wedge \cdots \wedge H_{A_d} = U_{1,E} \iff \{A_1, \ldots, A_d\} \textit{ satisfy } \mathrm{DHR}(M).$$

Thus, we have

$$\int_M h_{A_1}(M) \cdots h_{A_d}(M) = \begin{cases} 1 & \textit{if } \{A_1, \ldots, A_d\} \textit{ satisfy } \mathrm{DHR}(M), \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. For the first assertion, note that $M \wedge H_S = T_{\mathrm{cl}_M(S)}(M)$ has a loop iff $\mathrm{rk}_M(S) = 1$, and for the quotient $f : M \twoheadrightarrow M \wedge H_S$, by Proposition 3.13,

$$\{T \subseteq E : n_f(T) = 1\} = \{T \subseteq E : \mathrm{cl}_M(T) \supseteq S\}.$$

For the necessity of the condition, if $\mathrm{rk}_M\left(\bigcup_{j\in J} A_j\right) \leq k$ for some $J = \{j_1, \dots, j_k\} \subseteq \{1, \dots, d\}$, $k > 0$, then for $\widetilde{M} := M \wedge H_{A_{j_1}} \wedge \cdots \wedge H_{A_{j_{k-1}}}$ we have $\mathrm{rk}_{\widetilde{M}}\left(\bigcup_{j\in J} A_j\right) \leq k - (k-1) = 1$, so that $M \bigwedge_{j\in J} H_{A_j}$ already has a loop.
For sufficiency, we induct on d. The base case $d = 1$ is trivially satisfied. Now we claim that if $\{A_1, \dots, A_d\}$ satisfy the dragon Hall-Rado condition for M, then so does $\{A_1, \dots, A_{d-1}\}$ for $\widetilde{M} := M \wedge H_{A_d}$. For the sake of contradiction, suppose $\mathrm{rk}_{\widetilde{M}}(A_1 \cup \cdots \cup A_k) \leq k$, then we must have had $\mathrm{rk}_M(A_1 \cup \cdots \cup A_k) = k + 1$ with $\mathrm{cl}_M(A_1 \cup \cdots \cup A_k) \supseteq A_d$. But then $\mathrm{rk}_M(A_1 \cup \cdots \cup A_k \cup A_d) = k + 1$, violating $\mathrm{DHR}(M)$.
For the second assertion, note that $\int_M h_{A_1}(M) \cdots h_{A_d}(M) = \int_{\Sigma_{A_n}} h_{A_1} \cdots h_{A_d} \cap \Delta_M$, and the unique loopless matroid $U_{1,E}$ of rank 1 on E defines the Bergman class $\Delta_{U_{1,E}}$ by $\Delta_{U_{1,E}}(\mathbf{0}) = 1$, where $\mathbf{0}$ is the zero-dimensional cone of Σ_{A_n}, so that $\int_{\Sigma_{A_n}} \Delta_{U_{1,E}} = 1$. □

Corollary 4.10. *Let M be a loopless matroid on E of rank $d+1$. The volume polynomial* $\mathrm{VP}_M^{\nabla}(\mathbf{t}) \in \mathbb{R}[t_F : F \in \mathcal{L}_M^{\geq 2}]$ *of* $A_{\nabla}^{\bullet}(M)$ *is*

$$\mathrm{VP}_M^{\nabla}(\mathbf{t}) = \sum_{(F_1,\dots,F_d)} t_{F_1} \cdots t_{F_d}$$

where the sum is over ordered collections of nonempty flats $F_1, \dots, F_d$ satisfying $\mathrm{DHR}(M)$*. Alternatively, we have*

$$\mathrm{VP}_M^{\nabla}(\mathbf{t}) = \sum_{\{F_1^{d_1},\dots,F_k^{d_k}\}} \binom{d}{d_1, \dots, d_k} t_{F_1}^{d_1} \cdots t_{F_k}^{d_k}$$

where the sum is over size d multisets $\left\{F_1^{d_1}, \dots, F_k^{d_k}\right\}$ of flats of M satisfying $\mathrm{DHR}(M)$*.*

Example 4.11 ($U_{3,4}$, continued)**.** Consider the uniform matroid $M = U_{3,4}$ of rank 3 on $E = \{1, 2, 3, 4\}$. By the previous remark, we substitute the variables h_F of the defining ideal I of $A_{\nabla}^{\bullet}(M) = \mathbb{R}[h_1, h_2, h_3, h_4, h_{12}, \dots, h_{34}, h_{1234}]/I$

by the partial differential operators $\partial_F = \frac{\partial}{\partial h_F}$ and get a system

$$\begin{aligned}\partial_1\varphi = \partial_2\varphi = \partial_3\varphi = \partial_4\varphi = 0\\ (\partial_1 - \partial_{12} - \partial_{13} - \partial_{14} + 2\partial_{1234})(\partial_2 - \partial_{12} - \partial_{23} - \partial_{24} + 2\partial_{1234})\varphi = \cdots = 0\\ (\partial_{12} - \partial_{1234})(\partial_{13} - \partial_{1234})\varphi = \cdots = 0\\ (\partial_1 - \partial_{12} - \partial_{13} - \partial_{14} + 2\partial_{1234})(\partial_{23} - \partial_{1234})\varphi = \cdots = 0\end{aligned}$$

of 4+15+6+12=37 partial differential equations. Solving this system yields

$$\varphi = C_1\varphi_1 + C_2h_{12} + C_3h_{13} + C_4h_{14} + C_5h_{23} + C_6h_{24} + C_7h_{34} + C_8h_{1234} + C_9,$$

where $C_1, \dots, C_9 \in \mathbb{R}$ and

$$\varphi_1 = 2 \sum_{\substack{F,F' \in \mathcal{L}_M \setminus \{\emptyset\} \\ F \neq F'}} h_F h_{F'} + \sum_{F \in \mathcal{L}_M \setminus \{\emptyset\}} h_F^2 = \mathrm{VP}_M^{\nabla}$$

is the volume polynomial VP_M^{∇} of $A_{\nabla}^{\bullet}(M)$ (up to a constant factor). The solution space is a 9-dimensional space spanned by VP_M^{∇} and all of its partial derivatives, which is isomorphic to $A_{\nabla}^{\bullet}(M)$. It is easy to see that every pair (F_1, F_2) of flats F_1, F_2 of M of rank at least two satisfies $\mathrm{DHR}(M)$.

Remark. In [Eur20], an explicit formula for the volume polynomial VP_M of $A^{\bullet}(M)$ is given as

$$\begin{aligned}\mathrm{VP}_M(\mathbf{t}) = &\sum_{\{F_1^{d_1}, \dots, F_k^{d_k}\}} (-1)^{d-k} \binom{d}{d_1, \dots, d_k} \cdot \\ &\cdot \prod_{i=1}^{k} \binom{d_i - 1}{\sum_{j=1}^{i} d_j - \mathrm{rk}_M(F_i)} \mu^{\sum_{j=1}^{i} d_j - \mathrm{rk}_M(F_i)}(M|F_{i+1}/F_i) t_{F_1}^{d_1} \cdots t_{F_k}^{d_k},\end{aligned}$$

where the sum is over size d multisets $\left\{F_1^{d_1}, \dots, F_k^{d_k}\right\}$ of flats of M.

4.2.2 Volume polynomials of matroids are Lorentzian

Lorentzian polynomials were introduced and studied in [BH19]. The property to be Lorentzian is closely related to log-concavity. Lorentzian polynomials generalize stable polynomials and volume polynomials in algebraic geometry. Let $\{D_1, \dots, D_s\}$ be nef (ample) divisors on a projective $\mathbb{K}$-variety X of dimension d, and $A(X)$ its Chow ring. Let $\int_X : A^d(X) \to \mathbb{R}$ be the degree map obtained as the pushforward map along the structure map $X \to \operatorname{Spec} \mathbb{K}$. Then

$$\operatorname{vol}_X\left(\sum_{i=1}^{s} t_i D_i\right) := \lim_{q\to\infty} \frac{\dim_{\mathbb{K}} H^0(q\sum_i t_i D_i)}{q^d/d!} = \int_X \left(\sum_i t_i D_i\right)^d$$

is a (strictly) Lorentzian polynomial [BH19, Theorem 10.1]. As the simplicial generators of $A^\bullet_\nabla(M)$ are pullbacks of nef divisors on Σ_{A_n}, we are motivated to prove the volume polynomial of $A^\bullet_\nabla(M)$ to be Lorentzian.

Definition 4.4. A homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d is *strictly Lorentzian* if its support consists of all monomials in $\mathbf{x}$ of degree d, all of its coefficients are positive, and any of its $(d-2)$-th order partial differentiation $\partial_{i_1} \cdots \partial_{i_{d-2}} f$ has Hessian matrix with *Lorentzian signature* $(+, -, -, \ldots, -)$.

The set of strictly Lorentzian polynomials of degree d in n variables is denoted by $\mathring{\mathsf{L}}^d_n$. By the continuity of differential operators, the set $\mathring{\mathsf{L}}^d_n$ is open in the space of homogeneous degree d polynomials with respect to the product topology.

Definition 4.5. A collection of points $J \subset \mathbb{Z}^n_{\geq 0}$ is *M-convex* if for any $\alpha, \beta \in J$ and $i \in [n]$ with $\alpha_i > \beta_i$ there exists $j \in [n]$ such that $\alpha_j < \beta_j$ and $\alpha - \mathbf{e}_i + \mathbf{e}_j \in J$.

Definition 4.6. A homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d with non-negative coefficients is *Lorentzian* if

1. the support of f is M-convex, and
2. the Hessian of $\partial_{i_1} \cdots \partial_{i_{d-2}} f$ has at most one positive eigenvalue for any choice of $(d-2)$-th order partial differentiation.

We denote by L^d_n the set of Lorentzian polynomials $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d. The following theorem [BH19, Theorem 2.13] is proven by applying a sequence of operations preserving Lorentzian property.

Theorem 4.12. *The space $\mathring{\mathsf{L}}^d_n$ is contractible, and its closure contains L^d_n.*

It was shown in [BH19, Theorem 5.1] that the closure of $\mathring{\mathsf{L}}^d_n$ is exactly L^d_n. In other words, Lorentzian polynomials are polynomials that can be obtained as a limit of strictly Lorentzian polynomials. We show next that the volume polynomial VP^∇_M of a loopless matroid M is Lorentzian. First, we show that the dragon Hall-Rado condition description for the support of VP_M^{∇} implies it to be M-convex, then we compute the signatures of the Hessian matrices of partial derivatives of VP_M^{∇}.

Proposition 4.13. *Let $\{F_1, \dots, F_d\}$ and $\{G_1, \dots, G_d\}$ be two multisets of flats of a loopless matroid M such that both $t_{F_1} \cdots t_{F_d}$ and $t_{G_1} \cdots t_{G_d}$ are in the support of* VP_M^{∇}. *If (without loss of generality) G_d is a flat which appears more times in $\{G_1, \dots, G_d\}$ than it does in $\{F_1, \dots, F_d\}$, then there exists another flat F_m which appears more times in $\{F_1, \dots, F_d\}$ than it does in $\{G_1, \dots, G_d\}$ such that $t_{F_1} \cdots t_{F_d} t_{G_d}/t_{F_m}$ is in the support of* VP_M^{∇}.

Proof. We borrow standard language from polymatroid theory. A nonempty multiset of flats $\{A_1, \dots, A_k\}$ is called *dependent* if $\mathrm{rk}_M(\bigcup_{j=1}^k A_j) \leq k$. The condition to be in the support of VP_M^{∇}, the dragon Hall-Rado condition, is equivalent to the independence of every nonempty subset. We claim that the multiset of flats $\{F_1, \dots, F_d, G_d\}$ contains a unique minimally dependent multiset of flats X, which we call a *circuit*. Notice that $\{F_1, \dots, F_d, G_d\}$ is dependent, and any of its subset that is dependent contains G_d. The theorem will follow from this claim because the circuit X is not fully contained in $\{G_1, \dots, G_d\}$, hence we can let F_m be any flat in X which appears more times in $\{F_1, \dots, F_d\}$ than it does in $\{G_1, \dots, G_d\}$.
To prove the claim, support to the contrary that $\{R_1, \dots, R_a\}, \{S_1, \dots, S_b\}$ are two distinct circuits which are subsets of $\{F_1, \dots, F_d, G_d\}$. Let $\{T_1, \dots, T_c\} := \{R_1, \dots, R_a\} \cap \{S_1, \dots, S_b\}$. We claim that $\{T_1, \dots, T_c\}$ is dependent. Suppose to the contrary that $\mathrm{rk}_M(\bigcup_{j=1}^c T_j) \geq c+1$. Let $R := \bigcup_{j=1}^a R_j$ and $S := \bigcup_{j=1}^b S_j$. By assumption, we have $\mathrm{rk}_M(R) = a$ and $\mathrm{rk}_M(S) = b$. Submodularity gives that

$$\begin{aligned}\mathrm{rk}_M(R \cup S) &\leq \mathrm{rk}_M(R) + \mathrm{rk}_M(S) - \mathrm{rk}_M(R \cap S) = a + b - \mathrm{rk}_M(R \cap S)\\ &\leq a + b - \mathrm{rk}_M(\bigcup_{j=1}^c T_j) \leq a + b - c - 1.\end{aligned}$$

Without loss of generality, assume that $G_d = R_a = S_b = T_c$. We have that $R = \bigcup_{j=1}^{a-1} R_j$ and $S = \bigcup_{j=1}^{b-1} S_j$, otherwise $\{R_1, \dots, R_{a-1}\}$ and $\{S_1, \dots, S_{b-1}\}$ would both be dependent in $\{F_1, \dots, F_d\}$. Therefore, the union of elements in $\{R_1, \dots, R_{a-1}, S_1, \dots, S_{b-1}\} \backslash \{T_1, \dots, T_{c-1}\}$ is $R \cup S$ and

$$|\{R_1, \dots, R_{a-1}, S_1, \dots, S_{b-1}\} \backslash \{T_1, \dots, T_{c-1}\}| = (a-1)+(b-1)-(c-1) = a+b-c-1.$$

But we have calculated $\mathrm{rk}_M(R \cup S) \leq a + b - c - 1$, therefore the subset $\{R_1, \dots, R_{a-1}, S_1, \dots, S_{b-1}\} \backslash \{T_1, \dots, T_{c-1}\}$ of $\{F_1, \dots, F_d\}$ is dependent, contradiction. □

Theorem 4.14. *The volume polynomial* $\mathrm{VP}_M^{\nabla} \in \mathbb{R}[t_F : F \in \mathcal{L}_M^{\geq 2}]$ *of a loopless matroid* M *is Lorentzian.*

Proof. Let M be a loopless matroid of rank $r = d + 1$. There is nothing to prove if $d = 1$, so we assume $d \geq 2$. The support of VP_M^{∇} is M-convex by the previous proposition. Observe that for a flat G of rank at least 2, we have

$$\frac{\partial}{\partial t_G}\mathrm{VP}_M^{\nabla}(\mathbf{t}) = d\int_M h_G\Big(\sum_{F\in\mathcal{L}_M^{\geq 2}} t_F h_F\Big)^{d-1} = d\int_{T_G(M)}\Big(\sum_{F\in\mathcal{L}_M^{\geq 2}} t_F h_{\mathrm{cl}_{T_G(M)}(F)}\Big)^{d-1}.$$

Now, suppose $\{F_1, \ldots, F_{d-2}\}$ is a multiset of size $d-2$ consisting of flats of M with at least 2. We may assume that $\{F_1, \ldots, F_{d-2}\}$ satisfies $\mathrm{DHR}(M)$ because otherwise $\partial_{t_{F_1}} \cdots \partial_{t_{F_{d-2}}} \mathrm{VP}_M^{\nabla} \equiv 0$. One computes that

$$\partial_{t_{F_1}} \cdots \partial_{t_{F_{d-2}}} \mathrm{VP}_M(\mathbf{t}) = \frac{d!}{2}\int_{M'}\Big(\sum_{F\in\mathcal{L}_M^{\geq 2}} t_F h_{\mathrm{cl}_{M'}(F)}\Big)^2,$$

where $M' = M \wedge H_{F_1} \wedge \cdots \wedge H_{F_{d-2}}$ is a loopless matroid of rank 3. By [BH19, Theorem 2.10], if $f \in \mathbb{R}[x_1, \ldots, x_n]$ is Lorentzian, then so is $f(A\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_m]$ for any $n \times m$ matrix A with non-negative entries. So it suffices to check that $\mathrm{VP}_{M'}^{\nabla}$ is Lorentzian. For any loopless matroid M' of rank 3, the degree 1 part $A_{\nabla}^1(M)$ of its Chow ring has the simplicial basis $\{h_E\} \cup \{h_F : \mathrm{rk}_{M'}(F) = 2\}$. Noting that by Theorem 4.9, $\int_{M'} h_G h_{G'} = 1$ if $G \neq G'$ or $G = G' = E$, and 0 otherwise, the Hessian of the quadratic form $\mathrm{VP}_{M'}^{\nabla}$ is two time the matrix

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}, \text{ which reduces to } \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix}$$

by symmetric Gaußian elimination. □

4.2.3 Hodge-Riemann relations for Lorentzian polynomials

In [BH19, §4], it was proven that Lorentzian polynomials satisfy a formal version of the Hodge-Riemann relations. It provides a key step for the induction to show the Hodge-Riemann relations for the Chow rings of matroids.

First, we consider a subclass of Lorentzian polynomials.

Definition 4.7. Let $\mathscr{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ denote the open upper half of the complex plane. A polynomial $f \in \mathbb{R}[x_1, \cdots, x_n]$ is *stable* provided that either $f \equiv 0$ or $f(\mathbf{z}) \neq 0$ for any $\mathbf{z} \in \mathscr{H}^n$. We denote by S_n^d the set of degree d homogeneous stable polynomials in n variables with non-negative coefficients.

We refer to [BBL09; Wag11] for stable polynomials. It follows from Hurwitz's theorem in complex analysis that the set S_n^d is closed in the space of degree d homogeneous polynomials in n variables. We say polynomials in the interior $\mathring{\mathsf{S}}_n^d$ of S_n^d to be *strictly stable*. We denote by $\mathbf{H}_f := (\partial_i \partial_j f)_{i,j}$ the Hessian of f.

Lemma 4.15. *A polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is stable iff for all $\mathbf{v} \in \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^n_{>0}$, the univariable polynomial $f(\mathbf{v} + \mathbf{u}t)$ has only real roots.*

Proof. Since $\mathscr{H}^n = \{\mathbf{v} + \mathbf{u}t : \mathbf{v} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^n_{>0}, t \in \mathscr{H}\}$, the result follows. □

Lemma 4.16. *A homogeneous quadratic polynomial with positive coefficients is strictly Lorentzian iff it is strictly stable.*

Proof. Let f be a strictly Lorentzian quadratic polynomial. Since all entries of $\mathbf{H}_f$ are positive, $\mathbf{u}^\top \mathbf{H}_f \mathbf{u} > 0$ for any nonzero $\mathbf{u} \in \mathbb{R}^n_{\geq 0}$. By Cauchy's interlacing theorem, for any $\mathbf{v} \in \mathbb{R}^n$ not parallel to $\mathbf{u}$, the restriction of $\mathbf{H}_f$ to the plane spanned by $\mathbf{u}, \mathbf{v}$ has signature $(+, -)$. It follows that the discriminant of the univariable polynomial $\frac{1}{2} f(x\mathbf{u} - \mathbf{v})$ satisfies

$$(\mathbf{u}^\top \mathbf{H}_f \mathbf{v})^2 - (\mathbf{u}^\top \mathbf{H}_f \mathbf{u})(\mathbf{v}^\top \mathbf{H}_f \mathbf{v}) = -\det \begin{pmatrix} \mathbf{u}^\top \mathbf{H}_f \mathbf{u} & \mathbf{u}^\top \mathbf{H}_f \mathbf{v} \\ \mathbf{u}^\top \mathbf{H}_f \mathbf{v} & \mathbf{v}^\top \mathbf{H}_f \mathbf{v} \end{pmatrix} > 0,$$

hence $\frac{1}{2} f(x\mathbf{u} - \mathbf{v})$ has two distinct real roots, which is equivalent to the condition of being strictly Lorentzian.
Conversely, if the quadratic homogeneous polynomial f with positive coefficients is strictly stable, then there is some $\mathbf{u} \in \mathbb{R}^n_{>0}$ such that for any $\mathbf{v} \in \mathbb{R}^n$ not parallel to $\mathbf{u}$, the univariable polynomial $\frac{1}{2} f(x\mathbf{u} - \mathbf{v})$ has two distinct real roots, then $(\mathbf{u}^\top \mathbf{H}_f \mathbf{u})(\mathbf{v}^\top \mathbf{H}_f \mathbf{v}) - (\mathbf{u}^\top \mathbf{H}_f \mathbf{v})^2 < 0$. Therefore, $\mathbf{H}_f$ is negative definite on the hyperplane $\{\mathbf{v} \in \mathbb{R}^n : \mathbf{u}^\top \mathbf{H}_f \mathbf{v} = 0\}$. Since $\mathbf{u}^\top \mathbf{H}_f \mathbf{u} > 0$ as $\mathbf{u} \in \mathbb{R}^n_{>0}$, $\mathbf{H}_f$ has the Lorentz signature. □

Proposition 4.17. *Any polynomial in $\mathring{\mathsf{S}}_n^d$ is strictly Lorentzian.*

Proof. When $d = 2$, the statement follows from Lemma 4.16. In general, homogeneous strictly stable polynomials are strictly Lorentzian, since ∂_i is an open map sending S_n^d to S_n^{d-1} by the open mapping theorem in functional analysis. □

Now let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a nonzero degree $d \geq 2$ homogeneous polynomial with non-negative coefficients. The following is an analog of the Hodge-Riemann relations for homogeneous stable polynomials.

Proposition 4.18. *If f is in $\mathsf{S}_n^d \backslash \{0\}$, then $\mathbf{H}_f$ has exactly one positive eigenvalue for all $\mathbf{x} \in \mathbb{R}^n_{>0}$. Moreover, if f is in $\mathring{\mathsf{S}}_n^d$, then $\mathbf{H}_f(\mathbf{x})$ is nonsingular for all $\mathbf{x} \in \mathbb{R}^n_{>0}$.*

Proof. Fix a vector $\mathbf{x} \in \mathbb{R}^n_{>0}$. The quadratic polynomial

$$\mathbf{z}^\top \mathbf{H}_f(\mathbf{x})\mathbf{z} = \sum_{1 \leq i,j \leq n} z_i z_j \partial_i \partial_j f(\mathbf{x})$$

in $\mathbf{z}$ has Hessian $2\mathbf{H}_f(\mathbf{x})$. By Lemma 4.16 and the fact $\mathbf{1}^\top \mathbf{H}_f(\mathbf{x})\mathbf{1} > 0$, $\mathbf{H}_f(\mathbf{x})$ has exactly one positive eigenvalue if $\mathbf{z}^\top \mathbf{H}_f(\mathbf{x})\mathbf{z}$ is stable. One sees that $\mathbf{z}^\top \mathbf{H}_f(\mathbf{x})\mathbf{z}$ is the quadratic part of the stable polynomial $f(\mathbf{z}+\mathbf{x})$, hence it is stable because it can be obtained from $f(\mathbf{z}+\mathbf{x})$ by a sequence of homogenization, differentiation, inversion and specialization (see [BBL09, Lemma 4.16]), which are operations preserving stability, see [Wag11, Lemma 2.4] and [BBL09, Theorem 4.5].
Moreover, if f is strictly stable, then $f_\epsilon = f \pm \epsilon(x_1^d + \cdots + x_n^d)$ is stable for all sufficiently small positive ϵ. Therefore, by above, the matrix $\mathbf{H}_{f_\epsilon}(\mathbf{x}) = \mathbf{H}_f(\mathbf{x}) \pm d(d-1)\epsilon \mathrm{diag}(x_1^{d-2}, \ldots, x_n^{d-2})$ has exactly one positive eigenvalue for all sufficiently small $\epsilon > 0$, hence $\mathbf{H}_f(\mathbf{x})$ is nonsigular. □

The following is an analog of Lefschetz property for Lorentzian polynomials.

Lemma 4.19. *If $\mathbf{H}_{\partial_i f}(\mathbf{x})$ has exactly one positive eigenvalue for every $i \in [n]$ and $\mathbf{x} \in \mathbb{R}^n_{>0}$, then*

$$\ker \mathbf{H}_f(\mathbf{x}) = \bigcap_{i=1}^n \ker \mathbf{H}_{\partial_i f}(\mathbf{x}).$$

Proof. We may suppose $d \geq 3$. Fix $\mathbf{x} \in \mathbb{R}^n_{>0}$, and write $\mathbf{H}_f$ for $\mathbf{H}_f(\mathbf{x})$. By Euler's homogeneous function theorem,

$$(d-2)\mathbf{H}_f = \sum_{i=1}^n x_i \mathbf{H}_{\partial_i f}.$$

It follows that the kernel of $\mathbf{H}_f$ contains the intersection of the kernels of $\mathbf{H}_{\partial_i f}$.
For the other inclusion, let $\mathbf{z} \in \ker \mathbf{H}_f$. By Euler's homogeneous function theorem, $(d-2)\mathbf{e}_i^\top \mathbf{H}_f = \mathbf{x}^\top \mathbf{H}_{\partial_i f}$, hence $\mathbf{x}^\top \mathbf{H}_{\partial_i f}\mathbf{z} = 0$. We have $\mathbf{x}^\top \mathbf{H}_{\partial_i f}\mathbf{x} > 0$ because $\partial_i f$ is nonzero and has nonnegative coefficients. It follows that $\mathbf{H}_{\partial_i f}$ is negative on the kernel of $\mathbf{x}^\top \mathbf{H}_{\partial_i f}$. In particular,

$$\mathbf{z}^\top \mathbf{H}_{\partial_i f}\mathbf{z} \leq 0, \text{ with equality iff } \mathbf{H}_{\partial_i f}\mathbf{z} = 0.$$

To conclude, we write zero as the positive linear combination

$$0 = (d-2)\left(\mathbf{z}^\top \mathbf{H}_f \mathbf{z}\right) = \sum_{i=1}^{n} x_i \left(\mathbf{z}^\top \mathbf{H}_{\partial_i f}\mathbf{z}\right).$$

Since every summand in the right-hand side is non-positive, we must have $\mathbf{z}^\top \mathbf{H}_{\partial_i f}\mathbf{z} = 0$ for every i, and hence $\mathbf{H}_{\partial_i f}\mathbf{z} = 0$ for every i. □

We are now ready to generalize the Hodge-Riemann relations to Lorentzian polynomials.

Theorem 4.20. *Let f be a nonzero homogeneous polynomial in $\mathbb{R}[x_1, \ldots, x_n]$ of degree $d \geq 2$.*

(1) If f is strictly Lorentzian, then $\mathbf{H}_f(\mathbf{x})$ is nonsingular for all $\mathbf{x} \in \mathbb{R}^n_{>0}$.

(2) If f is Lorentzian, then $\mathbf{H}_f(\mathbf{x})$ has exactly one positive eigenvalue for $\mathbf{x} \in \mathbb{R}^n_{>0}$.

Proof. By Theorem 4.12, L^d_n is in the closure of $\mathring{\mathsf{L}}^d_n$. Therefore, we may suppose $f \in \mathring{\mathsf{L}}^d_n$ in (2). We prove (1) and (2) simultaneously by induction on d under this assumption. The base case $d = 2$ is trivial. We suppose that $d \geq 3$ and that theorem holds for $\mathring{\mathsf{L}}^{d-1}_n$.
That (1) holds for $\mathring{\mathsf{L}}^d_n$ follows from induction and Lemma 4.19. Using Proposition 4.18, we see that (2) holds for stable polynomials in $\mathring{\mathsf{L}}^d_n$. Since $\mathring{\mathsf{L}}^d_n$ is connected by Theorem 4.12, the continuity of eigenvalues and the validity of (1) together implies (2). □

4.3 Kähler Package in Degree at most One

4.3.1 Hard Lefschetz property and Hodge-Riemann relations

Definition 4.8. Let $(A^\bullet, \int)$ be a Poincaré duality $\mathbb{K}$-algebra of dimension d with degree map $\int$. For $\ell \in A^1$ and $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$, we define L^i_ℓ to be the *Lefschetz operator*

$$L^i_\ell : A^i \to A^{d-i}, \quad a \mapsto \ell^{d-2i} a,$$

and define Q^i_ℓ to be the *Hodge-Riemann symmetric bilinear form*

$$Q^i_\ell : A^i \times A^i \to \mathbb{K}, \quad (x, y) \mapsto \int xy\ell^{d-2i}.$$

We define the set of *degree i primitive classes of ℓ* to be

$$P^i_\ell := \left\{ x \in A^i : x\ell^{d-2i+1} = 0 \right\}.$$

Definition 4.9. Let $(A^\bullet, \int)$ be a Poincaré duality $\mathbb{R}$-algebra of dimension d and $\ell \in A^1$. For $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$, we say that $(A^\bullet, \int)$ satisfies

- (HL^i_ℓ) if L^i_ℓ induces an isomorphism between A^i and A^{d-i}, and
- (HR^i_ℓ) if the symmetric form $(-1)^i Q^i_\ell$ is positive definite when restrict to P^i_ℓ.

Moreover, for $\mathfrak{K}$ a convex cone in A^1, we say that $(A^\bullet, \int, \mathfrak{K})$ satisfies the *hard Lefschetz property* $(\mathrm{HL}^i_{\mathfrak{K}})$, resp. the *Hodge-Riemann relation* $(\mathrm{HR}^i_{\mathfrak{K}})$, in degree i if $A^\bullet$ satisfies (HL^i_ℓ), resp. (HR^i_ℓ), for all $\ell \in \mathfrak{K}$. We will write $(\mathrm{HL}^{\leq i})$ to mean hard Lefschetz property in degree at most i, and likewise for $(\mathrm{HR}^{\leq i})$. The properties (PD), (HL) and (HR) together are called the *Kähler package* for a graded ring $A^\bullet$.

Proposition 4.21. *Let $(A^\bullet, \int, \mathfrak{K})$ be a Poincaré duality algebra which satisfies $(\mathrm{HL}^i_{\mathfrak{K}})$ for a convex cone $\mathfrak{K}$ in A^1. Suppose that $(A^\bullet, \int)$ satisfies (HR^i_ℓ) for some $\ell \in \mathfrak{K}$. Then $A^\bullet$ satisfies $(\mathrm{HR}^i_{\mathfrak{K}})$.*

Proof. Let $\ell' \in \mathfrak{K}$, and $l(t) = t\ell + (1-t)\ell'$ for $t \in [0,1]$ be a line segment connecting ℓ and ℓ'. By convexity of $\mathfrak{K}$, every point on l is in $\mathfrak{K}$. If the signature of the bilinear pairing $Q^i_{l(t)}$ changes along $l(t)$ starting at ℓ, then it must degenerate at some point $l(t_0)$ for $t_0 \in [0,1]$, but this violates $(\mathrm{HL}^i_{\mathfrak{K}})$. $\square$

It is easy to check that the tensor product of Poincaré duality algebras is also a Poincaré duality algebra. Remark that this is the analogue of the Künnuth formula in algebraic topology.

Proposition 4.22. *Let $(A^\bullet, \int_A)$ and $(B^\bullet, \int_B)$ be Poincaré duality algebras of dimension d_A and d_B over a common field $\mathbb{K}$. Then their tensor product*

$$(A \otimes B)^\bullet = \bigoplus_\bullet \left(\bigoplus_{i+j=\bullet} A^i \otimes B^j \right)$$

is a Poincaré duality algebra of dimension $d_A + d_B$ with degree map

$$\int_{A \otimes B} : (A \otimes B)^{d_A + d_B} = A^{d_A} \otimes B^{d_B} \to \mathbb{K}, \;\; a \otimes b \mapsto \int_A a \cdot \int_B b.$$

We now show how the properties (HL) and (HR) behave under tensor products and transports.

Proposition 4.23. *Let $(A^\bullet, \int_A)$ and $(B^\bullet, \int_B)$ be Poincaré duality algebras of dimension $d_A \geq 1$ and $d_B \geq 1$. Suppose that $A^\bullet$ and $B^\bullet$ satisfy $(\mathrm{HR}^{\leq 1}_{\ell_A})$ and $(\mathrm{HR}^{\leq 1}_{\ell_B})$, respectively, then $((A \otimes B)^\bullet, \int_{A \otimes B})$ satisfies $(\mathrm{HR}^{\leq 1}_{\ell_A \otimes 1 + 1 \otimes \ell_B})$.*

Proof. Set $\ell := \ell_A \otimes 1 + 1 \otimes \ell_B$ and $d := d_A + d_B$. First, by the properties $(\mathrm{HR}^0_{\ell_A})$ and $(\mathrm{HR}^0_{\ell_B})$ of $A^\bullet$ and $B^\bullet$, respectively, we have $\int_A \ell_A^{d_A} > 0$ and $\int_B \ell_B^{d_B} > 0$, therefore,

$$\int_{A \otimes B} \ell^d = \int_{A \otimes B} \sum_{k=1}^{d} \binom{d}{k} \left(\ell_A^k \otimes \ell_B^{d-k} \right) = \binom{d}{d_A} \int_A \ell_A^{d_A} \cdot \int_B \ell_B^{d_B} > 0,$$

hence $(A \otimes B)^\bullet$ satisfies (HR^0_ℓ).
Now let $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ be orthonormal bases for $P^1_{\ell_A}$ and $P^1_{\ell_B}$, respectively. Then

$$A^1 \cong \langle \ell_A \rangle \oplus \bigoplus_{i=1}^{m} \langle v_i \rangle \quad \text{and} \quad B^1 \cong \langle \ell_B \rangle \oplus \bigoplus_{i=1}^{n} \langle w_i \rangle.$$

Noting that $(A \otimes B)^\bullet$ is a Poincaré duality algebra of dimension d, we expand

$$\ell^{d-2} = (\ell_A \otimes 1 + 1 \otimes \ell_B)^{d-2} = \sum_{i=0}^{d-2} \binom{d-2}{i} \left(\ell_A^i \otimes \ell_B^{d-i-2} \right).$$

The symmetric matrix for Q^1_ℓ with respect to the above basis is given by

$$Q^1_\ell(a,b) = \begin{cases} -\binom{d-2}{d_A} & a = b = v_i \otimes 1 \\ -\binom{d-2}{d_B} & a = b = 1 \otimes w_j \\ \lambda\binom{d-2}{d_B} & a = b = \ell_A \otimes 1 \\ \lambda\binom{d-2}{d_A} & a = b = 1 \otimes \ell_B \\ \lambda\binom{d-2}{d_A-1} & a = \ell_A \otimes 1, b = 1 \otimes \ell_B \text{ or } a = 1 \otimes \ell_B, b = \ell_A \otimes 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda := \int_A \ell_A^{d_A} \cdot \int_B \ell_B^{d_B}$. So the matrix $Q^1_\ell(a,b)$ is a block matrix comprised of 3 blocks. By $(\mathrm{HR}^1_{\ell_A})$ and $(\mathrm{HR}^1_{\ell_B})$, the first two blocks are negative identity matrices induced by $\{v_i \otimes 1\} \times \{v_i \otimes 1\}$ and $\{1 \otimes w_j\} \times \{1 \otimes w_j\}$. The third and only nontrivial block is induced by $\{\ell_A \otimes 1, 1 \otimes \ell_B\} \times \{\ell_A \otimes 1, 1 \otimes \ell_B\}$, which gives the 2×2 matrix

$$M = \lambda \begin{bmatrix} \binom{d-2}{d_A-2} & \binom{d-2}{d_A-1} \\ \binom{d-2}{d_A-1} & \binom{d-2}{d_A} \end{bmatrix}.$$

One see from the log-concavity of binomial coefficients that $\det(M) < 0$, and hence M has signature $(+,-)$. We conclude that $Q^1_\ell(a,b)$ is nondegenerate and has exactly one positive eigenvalue completing the proof. $\square$

Proposition 4.24. *Let $(A^\bullet = \mathbb{R}[x_1,\ldots,x_s]/I, \int)$ be a Poincaré duality algebra of dimension d, and let $\ell \in A^1$ be a positive linear combination of $\{x_1,\ldots,x_s\}$. Denote by ℓ_k the image of ℓ in $A^\bullet/\mathrm{Ann}(x_k)$. For $0 \le i \le \lfloor \frac{d-1}{2} \rfloor$, if $(A^\bullet/\mathrm{Ann}(x_k), \int_{x_k})$ satisfies $(\mathrm{HR}^i_{\ell_k})$ for every $k = 1,\ldots,s$, then $(A^\bullet, \int)$ satisfies (HL^i_ℓ).*

Proof. Let $\ell = \sum_{k=1}^s c_k x_k$ with $c_k \in \mathbb{R}_{>0}$, and suppose $\ell^{d-2i} f = 0$ for some $f \in A^i$. We will show that $f = 0$. Let f_k be the image of f in $A^\bullet/\mathrm{Ann}(x_k)$. As $\ell^{d-2i} f = 0$, we have $\ell_k^{d-2i} f_k = 0$. Because $A^\bullet/\mathrm{Ann}(x_k)$ is a Poincaré duality algebra of dimension $d-1$, we conclude that $f_k \in P^i_{\ell_k}$. By the definition of $\int_{x_k}$, we have

$$0 = \int \ell^{d-2i} f^2 = \int \left(\sum_k c_k x_k\right) \ell^{d-2i-1} f^2 = \sum_k c_k \left(\int_{x_k} \ell_k^{d-2i-1} f_k^2\right),$$

where $(-1)^i \int_{x_k} \ell_k^{d-2i-1} f_k^2 \ge 0 \, \forall k$ by $(\mathrm{HR}^i_{\ell_k})$. Moreover, as $(-1)^i Q^i_{\ell_k}$ is positive definite on $P^i_{\ell_k}$, we conclude each f_k to be 0, that is, $x_k f = 0$ for all $k = 1,\ldots,s$. Since $\{x_1,\ldots,x_s\}$ generate $A^\bullet$, the Poincaré duality property of $A^\bullet$ implies that if $f \neq 0$ then there exists a polynomial $g(x)$ of degree $d-i$ such that $\int g(x) f \neq 0$, and hence we conclude that $f = 0$. $\square$

Let Σ be a d-dimensional smooth rational fan in $N_{\mathbb{R}}$ for a lattice N, and let $\rho \in \Sigma(1)$ be a ray. Denote by $\operatorname{star}(\rho,\Sigma)_\rho$ the image of $\operatorname{star}(\rho,\Sigma)$ under the projection $N_{\mathbb{R}} \twoheadrightarrow N_{\mathbb{R}}/\operatorname{span}(\rho)$, which is a $(d-1)$-dimensional fan in $N_{\mathbb{R}}/\operatorname{span}(\rho)$. By definition of the Chow ring, there is a surjection

$$A^\bullet(\Sigma) \twoheadrightarrow A^\bullet(\operatorname{star}(\rho,\Sigma)_\rho), \quad x_{\rho'} \mapsto \begin{cases} x_{\overline{\rho'}} & \text{if } \rho' \text{ and } \rho \text{ form a cone in } \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

for each $\rho' \neq \rho$, where $\overline{\rho'}$ is the image of ρ' under the projection $N_{\mathbb{R}} \twoheadrightarrow N_{\mathbb{R}}/\operatorname{span}(\rho)$. As $\langle x_{\rho'} \mid \rho' \text{ and } \rho \text{ do not form a cone in } \Sigma\rangle \subseteq \operatorname{Ann}_{A^\bullet(\Sigma)}(x_\rho)$, we get an induced map

$$\pi_\rho : A^\bullet(\operatorname{star}(\rho,\Sigma)_\rho) \twoheadrightarrow A^\bullet(\Sigma)/\operatorname{Ann}(x_\rho).$$

We have the following criterion for when the map π_ρ is an isomorphism.

Proposition 4.25. *Suppose Σ satisfies $\mathrm{MW}_d(\Sigma) \cong \mathbb{R}$, and suppose further that the Chow ring $A^\bullet(\Sigma)$ is a Poincaré duality algebra of dimension d with the degree map $\int_\Sigma$ (for any choice of the fundamental class Δ_Σ). Then the map $\pi_\rho : A^\bullet(\operatorname{star}(\rho,\Sigma)_\rho) \twoheadrightarrow A^\bullet(\Sigma)/\operatorname{Ann}(x_\rho)$ is an isomorphism iff $A^\bullet(\operatorname{star}(\rho,\Sigma)_\rho)$ is a Poincaré duality algebra.*

Proof. For the "if" part note that a surjective map of Poincaré duality algebras of the same dimension is an isomorphism: If $\varphi : A^\bullet \twoheadrightarrow B^\bullet$ is a surjection of Poincaré duality algebras of the same dimension d, then $\ker\varphi \neq \langle 0\rangle$ implies that $\ker\varphi \supseteq A^d$, so that $B^\bullet \cong A^\bullet/I$ is a Poincaré duality algebra of dimension at most $d-1$.
The "only if" part follows from Proposition 4.1. □

4.3.2 Kähler package in degree at most one for matroids

Now we will show that the Chow rings of matroids satisfy the hard Lefschetz property and the Hodge-Riemann relations in degree at most 1, which is the key part to prove the Heron-Rota-Welsh conjecture. Recall that we have the pullback map $\iota_M^* : A^\bullet(\Sigma_{A_n}) \twoheadrightarrow A^\bullet(M)$ via $x_S \mapsto x_S$ if S is a proper flat of M and 0 otherwise.

Definition 4.10. Let M be a loopless matroid on E. a divisor $D \in A^1(M)$ is *combinatorially nef* (resp. *ample*) if it is a pullback of a nef (ample) divisor on Σ_{A_n}. Explicitly, by Theorem 3.8, a divisor $D = \sum_{F \in \mathcal{L}_M \setminus \{\emptyset, E\}} c_F x_F$ is combinatorially nef (resp. ample) if there exists a function $a_{(\cdot)} : 2^E \to \mathbb{Z}$ with $a_\emptyset = a_E = 0$ such that $a_F = c_F$ $\forall F \in \mathcal{L}_M$ and

$$a_A + a_B \geq a_{A \cup B} + a_{A \cap B} \quad \text{for every } A, B \subseteq E$$

(resp. with strict inequality whenever A, B incomparable).
The nef (ample) divisors form a cone in $A^1(M)$ called the *combinatorial ample (nef) cone*, denoted by $\overline{\mathfrak{K}}_M$ ($\mathfrak{K}_M$).

Proposition 4.26. *A combinatorially nef divisor $D \in A^1(M)$ is effective. In particular, a combinatorially ample divisor $D \in A^1(M)$ can be written as a positive linear combination of x_F, $F \in \mathcal{L}_M \setminus \{\emptyset, E\}$.*

Proof. As a combinatorially nef (ample) divisor $D \in A^1(M)$ is a pullback of a nef (ample) divisor on Σ_{A_n}, this statement follows from Corollary 3.4. $\square$

Let $\mathfrak{K}_M^\nabla$ be the interior of the cone generated by the simplicial generators of $A_\nabla^\bullet(M)$. It is a subcone of $\mathfrak{K}_M$ since the simplicial generators are nef by construction. In light of Proposition 4.21, we need to establish (HR_ℓ^1) for some divisor $\ell \in \mathfrak{K}_M$ in order to prove the property $(\mathrm{HR}_{\mathfrak{K}_M}^1)$ of $A^\bullet(M)$. This is provided by the Hodge-Riemann relations of the Lorentzian polynomial $\mathrm{VP}_M^\nabla(\mathbf{t})$.

Lemma 4.27. *Let M be a loopless matroid of rank $r = d+1 \geq 2$. For any $\ell \in \mathfrak{K}_M^\nabla$, we have $\int_M \ell^d > 0$, and when $r = d+1 \geq 3$, the form Q_ℓ^1 has exactly one positive eigenvalue.*

Proof. The statement $\int_M \ell^d > 0$ follows from the dragon Hall-Rado formula in Corollary 4.10. The second statement follows from Theorem 4.14 and Theorem 4.20, because for $\ell = \sum t_i h_i$, the matrix of Q_ℓ^1 is $d(d-1)$ times the Hessian of $\mathrm{VP}_M^\nabla(\mathbf{t})$. $\square$

The following lemma provides the key step to reduce the rank in the induction. It underlies the Hopf algebraic structure for the lattice of flats of a matroid.

Lemma 4.28. *Let M be a loopless matroid, and F a nonempty proper flat of M. Let ρ_F be the ray corresponding to F in the Bergman fan Σ_M of M. We have*

$$\mathrm{star}(\rho_F, \Sigma_M)_{\rho_F} \cong \Sigma_{M|F} \times \Sigma_{M/F}, \tag{4.3}$$

and consequently an isomorphism of Poincaré duality algebras

$$A^\bullet(M)/\mathrm{Ann}(x_F) \cong (A(M|F) \otimes A(M/F))^\bullet \tag{4.4}$$

such that if $\ell \in \mathfrak{K}_M$ then its image in $A^\bullet(M)/\mathrm{Ann}(x_F)$ is in $(\mathfrak{K}_{M|F} \otimes 1) \oplus (1 \otimes \mathfrak{K}_{M/F})$.

Proof. A cone of Σ_M is in $\mathrm{star}(\rho_F, \Sigma_M)$ iff it corresponds to a flag containing F. By Corollary 2.4, any such flag naturally factors as the concatenation of two flags, one with maximal element strictly contained in F, and the other with minimal element F. This geometrically corresponds to the factorization of fans in (4.3). For (4.4), first note that $M|F$ and M/F are loopless since F is a flat. Then (4.4) follows from Proposition 4.22 and Proposition 4.25 and the fact that $A^\bullet(\Sigma \times \Sigma') \cong (A(\Sigma) \otimes A(\Sigma'))^\bullet$ for rational fans Σ, Σ'. The last statement follows from the fact that the restriction of submodular functions on lattice remain submodular under restriction to intervals in the lattice. □

Theorem 4.29. *The Chow ring of a matroid $(A^\bullet(M), \int_M, \mathfrak{K}_M)$ satisfies $(\mathrm{HL}^{\leq 1}_{\mathfrak{K}_M})$ and $(\mathrm{HR}^{\leq 1}_{\mathfrak{K}_M})$.*

Proof. We proceed by induction on the rank of the matroid M. For rank 1 matroids, the stated properties are trivially satisfied. Let M be a rank 2 matroid. By Proposition 4.26, any nef divisor $\ell \in \mathfrak{K}_M$ is effective, that is, there are non-negative numbers $c_a \geq 0$ such that $\ell = \sum_{a \in \mathfrak{A}(M)} c_a x_a$. Then $\int_M \ell = \sum_{a \in \mathfrak{A}(M)} c_a > 0$, which implies $(\mathrm{HL}^0_{\mathfrak{K}_M})$ and $(\mathrm{HR}^0_{\mathfrak{K}_M})$ for M.

Let M now be a loopless matroid of rank $r = d + 1 \geq 3$ on a ground set E. Observe that the property $(\mathrm{HR}^0_{\mathfrak{K}_M})$ holds iff $\int_M \ell^d > 0$ for all $\ell \in \mathfrak{K}_M$, and it implies $(\mathrm{HL}^0_{\mathfrak{K}_M})$. And given (HL^1_ℓ) and $\int_M \ell^d > 0$, the property (HR^1_ℓ) holds iff Q^1_ℓ has exactly one positive eigenvalue. Combined with Lemma 4.27 and Proposition 4.21, these facts imply that proving $(\mathrm{HL}^{\leq 1}_{\mathfrak{K}_M})$ is sufficient to establish $(\mathrm{HR}^{\leq 1}_{\mathfrak{K}_M})$. By Proposition 4.26, any element $\ell \in \mathfrak{K}_M$ can be written as a positive linear combination of $\{x_F : F \in \mathcal{L}_M \backslash \{\emptyset, E\}\}$. Therefore, by Proposition 4.24, to establish $(\mathrm{HL}^{\leq 1}_{\mathfrak{K}_M})$ (except $(\mathrm{HL}^1_{\mathfrak{K}_M})$ in the case of $\mathrm{rk}(M) = 3$, which is satisfied as L^1_ℓ is the identity function for any ℓ), it suffices in turn to prove $(\mathrm{HR}^{\leq 1}_{\mathfrak{K}_M})$ for $A^\bullet(M)/\mathrm{Ann}(x_F)$ for every nonempty proper flat F. Finally, $A^\bullet(M)/\mathrm{Ann}(x_F) \cong (A(M|F) \otimes A(M/F))^\bullet$ by Lemma 4.28, so by the induction hypothesis and Proposition 4.23, the proof is complete. □

4.4 Log-Concavity

Let M be a loopless matroid of rank $r = d + 1$ on $E = \{0, \ldots, n\}$. Recall that the characteristic polynomial of M is

$$\chi_M(q) = \sum_{F \in \mathcal{L}_M} \mu_M(\emptyset, F) q^{r-\mathrm{rk}(F)} = \sum_{k=0}^{r} w_k(M) q^{r-k},$$

where the Whitney numbers $w_k(M)$ satisfy $(-1)^k w_k(M) > 0$.
Because $\chi_M(1) = \sum_{F \in \mathcal{L}_M} \mu_M(\emptyset, F) = 0$, we define the *reduced characteristic polynomial* of M to be

$$\overline{\chi}_M(q) := \chi_M(q)/(q-1).$$

We define a sequence of integers $\mu^0(M), \ldots, \mu^{r-1}(M)$ by the equality

$$\overline{\chi}_M(q) = \sum_{k=0}^{r-1} (-1)^k \mu^k(M) q^{r-k}.$$

By comparing the coefficients, we have

$$|w_k(M)| = (-1)^k w_k(M) = \mu^k(M) + \mu^{k-1}(M) \text{ for } 0 \leq k \leq r$$

by convention $\mu^{-1}(M) = \mu^r(M) = 0$, and

$$\mu^k(M) = (-1)^k (w_0(M) + w_1(M) + \cdots + w_k(M)).$$

We will show that the Kähler package of $A^\bullet(M)$ in degree at most 1 implies the log-concavity of $\mu^k(M)$. Because the convolution of two log-concave sequences is log-concave, we have then the log-concavity of $|w_k(M)|$.

The following lattice theoretic theorem can be found e.g. in [Sta11, Corollary 3.9.3].

Lemma 4.30 (Weisner's theorem). *Let F be a flat of a loopless matroid M, and let $i \in F$ be an element. Then*

$$\mu_M(\emptyset, F) = -\sum_{a \notin F' \lessdot F} \mu_M(\emptyset, F'),$$

where $F' \lessdot F$ denotes that the flat $F' \in \mathcal{L}_M$ is covered by F, i.e. $F' \subseteq F$ and $\mathrm{rk}_M(F) = \mathrm{rk}_M(F') + 1$.

Lemma 4.31. *Let $a \in E$. The coefficients of $\overline{\chi}_M(q)$ are given by*

$$\mu^k(M) = (-1)^k \sum_{\substack{a \notin F \in \mathcal{L}_M \\ \mathrm{rk}_M(F)=k}} \mu_M(\emptyset, F) = (-1)^{k+1} \sum_{\substack{a \in F \in \mathcal{L}_M \\ \mathrm{rk}_M(F)=k+1}} \mu_M(\emptyset, F). \tag{4.5}$$

Proof. We begin by proving the second equality by applying Weisner's theorem

$$\sum_{\substack{a \in F \in \mathcal{L}_M \\ \mathrm{rk}_M(F)=k+1}} \mu_M(\emptyset, F) = - \sum_{\substack{a \in F \in \mathcal{L}_M \\ \mathrm{rk}_M(F)=k+1}} \sum_{a \notin F' \lessdot F} \mu_M(\emptyset, F')$$

$$= - \sum_{\substack{a \notin F \in \mathcal{L}_M \\ \mathrm{rk}_M(F)=k}} \sum_{a \in F \gtrdot F'} \mu_M(\emptyset, F') = - \sum_{\substack{a \notin F' \in \mathcal{L}_M \\ \mathrm{rk}_M(F')=k}} \mu_M(\emptyset, F')$$

where the last equality follows from the fact that for F' not containing a, there is a unique flat F with $\mathrm{rk}_M(F) = \mathrm{rk}_M(F') + 1$ and $a \in F$.
The theorem is true for $k = 0$. It now follows by induction using

$$\sum_{\substack{F \in \mathcal{L}_M \\ \mathrm{rk}_M(F)=k}} \mu_M(\emptyset, F) = (-1)^k w_k(M) = \mu^k(M) + \mu^{k-1}(M).$$

□

Now, we give a combinatorial interpretation for the coefficients $\mu^k(M)$.

Definition 4.11. Let $\mathcal{F} = (\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E)$ be a k-step flag of nonempty proper flats of M.

- The flag $\mathcal{F}$ is *initial* if $\mathrm{rk}_M(F_m) = m$ for all indices m.
- The flag $\mathcal{F}$ is *descending* if $\min(F_1) > \cdots > \min(F_k) > 0$.

We write $D_k(M)$ for the set of initial descending k-step flags of nonempty proper flats of M.

Theorem 4.32. *For every positive integer $k < r$, we have $\mu^k(M) = |D_k(M)|$.*

Proof. By iterative application of Weisner's Theorem on (4.5), we have

$$\mu^k = (-1)^k \sum_{\substack{0 \notin F_k \in \mathcal{L}_M \\ \mathrm{rk}_M(F_k)=k}} \mu_M(\emptyset, F_k) = (-1)^{k-1} \sum_{\substack{a \notin F_k \in \mathcal{L}_M \\ \mathrm{rk}_M(F_k)=k}} \sum_{\min F_k \notin F_{k-1} \lessdot F_k} \mu_M(\emptyset, F_{k-1})$$

$$= \cdots = - \sum_{\substack{a \notin F_k \in \mathcal{L}_M \\ \mathrm{rk}_M(F_k)=k}} \sum_{\min F_k \notin F_{k-1} \lessdot F_k} \cdots \sum_{\min F_2 \notin F_1 \lessdot F_2} (-1) = |D_k(M)|.$$

□

For any $i \in E$, we denote

$$\alpha := \sum_{i\in F} x_F \in A^1(M), \qquad \beta := \sum_{i\notin F} x_F \in A^1(M).$$

Both α and β are independent of the choice of i. For a flag $\mathcal{F} = (\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E)$ of flats in M we write $x_{\mathcal{F}} := x_{F_1} \cdots x_{F_k}$.

The elements $\alpha, \beta \in A^1(M)$ are nef: For a representation $\alpha = \sum_{i\in F} x_F$, the function $a_{(\cdot)} : 2^E \to \mathbb{Z}$, $a_F = 1$ if $i \in F \neq E$ and 0 otherwise is easy to check to be submodular and α is the pullback of the nef divisor $\sum a_F x_F$ on $A^\bullet(\Sigma_{A_n})$. For $\beta = \sum_{i\notin F} x_F$, it is the pullback of the nef divisor $\sum b_F x_F$ on $A^\bullet(\Sigma_{A_n})$ where $b_F = 1$ if $i \notin F$ and 0 otherwise.

Lemma 4.33. *Let $\mathcal{F} = (\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E)$ be any flag of nonempty proper flats of M. If the flag $\mathcal{F}$ is initial, then*

$$x_{\mathcal{F}}\alpha^{d-k} = \alpha^d \in A^d(M).$$

If $\mathcal{F}$ is not initial, then $x_{\mathcal{F}}\alpha^{d-k} = 0$.

Proof. First note that for any element i not in a nonempty proper flat F,

$$x_F\alpha = x_F \sum_{G\supseteq F\cup\{i\}} x_G \in A^\bullet(M).$$

If the flag $\mathcal{F}$ is not initial, we prove by descending induction on k. If $k = d-1$, then $\mathrm{rk}_M(F_k) = d$ so the product is zero. For general k, choose an element $i \notin F_k$. Then

$$x_{F_1} \cdots x_{F_k}\alpha^{d-k} = x_{F_1} \cdots x_{F_k} \left(\sum_{G\supseteq F_k\cup\{i\}} x_G \right) \alpha^{d-(k+1)}$$

which is zero by the induction hypothesis for $k+1$ applied to each of the terms in the expansion.

For an initial flag $\mathcal{F}$, we prove by ascending induction on k. When $k = 1$, the flat $F_1 \in \mathfrak{A}(M)$ is an atom. Choose an element $i \in F_1$ and we have

$$\alpha^d = \left(\sum_{i\in G} x_G \right) \alpha^{d-1} = x_{F_1}\alpha^{d-1}.$$

For general k, choose an element $i \in F_k \backslash F_{k-1}$. By the induction hypothesis,

$$\alpha^d = x_{F_1} \cdots x_{F_{k-1}} \alpha^{d-(k-1)} = x_{F_1} \cdots x_{F_{k-1}} \left(\sum_{G \supseteq F_{k-1} \cup \{i\}} x_G \right) \alpha^{d-k}.$$

For any G other than F_k, the flag $F_1 \subsetneq \cdots \subsetneq F_{k-1} \subsetneq F_G$ is not initial, hence it contributes a zero in the sum. Therefore, we get $\alpha^d = x_{F_1} \cdots x_{F_k} \alpha^{d-k}$. □

In particular, $\alpha^d = x_{F_1} \cdots x_{F_d}$ for any complete flag of nonempty proper flats $F_1 \subsetneq \cdots \subsetneq F_d$ and $\int_M \alpha^d = 1$.

Lemma 4.34. *For every positive integer $k \leq d$, we have*

$$\beta^k = \sum_{\mathcal{F}} x_{\mathcal{F}} \in A^\bullet(M),$$

where the sum is over all descending k-step flags $\mathcal{F}$ of nonempty proper flats of M.

Proof. We prove by induction on k. When $k = 1$, $\sum_{0 \notin F} x_F = \beta$ by the definition. In the general case, we use the induction hypothesis for k to write

$$\beta^{k+1} = \sum_{\mathcal{F}} \beta x_{\mathcal{F}},$$

where the sum is over all descending k-step flags $\mathcal{F}$ of nonempty proper flats of M. For each of the summands $\beta x_{\mathcal{F}}$, write $\mathcal{F} = (F_1 \subsetneq \cdots \subsetneq F_k)$ and set $i_{\mathcal{F}} := \min F_1$.

$$\beta x_{\mathcal{F}} = \left(\sum_{i_{\mathcal{F}} \notin F} x_F \right) x_{\mathcal{F}} = \sum_{\mathcal{F}'} x_{\mathcal{F}'}$$

where the last sum is over all descending flags of nonempty proper flats of M of the form $\mathcal{F}' = (F \subsetneq F_1 \subsetneq \cdots \subsetneq F_k)$. This completes the induction. □

Combining Theorem 4.32, Lemma 4.33 and Lemma 4.34, we see that the coefficients of the reduced characteristic polynomial of M are given by the degrees of the products $\alpha^{d-k}\beta^k$.

Theorem 4.35. *For every $0 \leq k \leq d$, we have*

$$\mu^k(M) = \int_M \alpha^{d-k} \beta^k.$$

Now we can apply the hard Lefschetz property and Hodge-Riemann relations in degree at most 1 to prove the log-concavity of $\mu^k(M)$, and hence the log-concavity of $w_k(M)$ and $f_k(M)$.

Lemma 4.36. *Let $\ell_1, \ell_2 \in A^1(M)$. If ℓ_2 is nef, then*

$$\left(\int_M \ell_1^2 \ell_2^{d-2}\right)\left(\int_M \ell_2^2 \ell_2^{d-2}\right) \leq \left(\int_M \ell_1 \ell_2 \ell_2^{d-2}\right)^2.$$

Proof. Suppose first that ℓ_2 is ample. By Theorem 4.29, $A^\bullet(M)$ satisfies $(\mathrm{HL}_{\ell_2}^{\leq 1})$, so we obtain a decomposition $A^1(M) \cong \langle \ell_2 \rangle \oplus P^1_{\ell_2}$ that is orthogonal with respect to the Hodge-Riemann form $Q^1_{\ell_2}$. By $(\mathrm{HR}_{\ell_2}^{\leq 1})$, $Q^1_{\ell_2}$ is negative definite on $P^1_{\ell_2}$ and positive definite on $\langle \ell_2 \rangle$. Therefore, the restriction of $Q^1_{\ell_2}$ to the subspace $\langle \ell_1, \ell_2 \rangle \subseteq A^1(M)$ is neither positive nor negative definite, so

$$\left(\int_M \ell_1^2 \ell_2^{d-2}\right)\left(\int_M \ell_2^2 \ell_2^{d-2}\right) < 0 \leq \left(\int_M \ell_1 \ell_2 \ell_2^{d-2}\right)^2.$$

If ℓ_2 is merely nef rather than ample, then for any ample element ℓ, the class $\ell_2(t) := \ell_2 + t\ell$ is ample for all $t > 0$. An ample ℓ exists as the subcone $\mathfrak{K}_M^{\nabla} \subseteq \mathfrak{K}_M$ is nonempty. Now, taking a limit as $t \to 0$ in the inequality

$$\left(\int_M \ell_1^2 \ell_2(t)^{d-2}\right)\left(\int_M \ell_2(t)^2 \ell_2(t)^{d-2}\right) < \left(\int_M \ell_1 \ell_2(t) \ell_2(t)^{d-2}\right)^2$$

yields the desired inequality. □

Corollary 4.37. *For each $0 < k < d$,*

$$\mu^{k-1}(M)\mu^{k+1}(M) \leq \mu^k(M)^2.$$

Proof. We prove by induction on $\mathrm{rk}(M)$. When $k < d-1$, consider the truncation $T_E(M)$. Recall that the lattice of flats of $T_E(M)$ is obtained from $\mathcal{L}_M$ by removing all flats of rank d. Therefore $D_l(M) = D_l(T_E(M))$, and by Theorem 4.32, $\mu^l(M) = \mu^l(T_E(M))$ for all $l < d$. The induction hypothesis applied to $T_E(M)$ implies the inequality.
Now, consider $k = d-1$. By Theorem 4.35, $\mu^k(M) = \int_M \alpha^{d-k}\beta^k$. Therefore, the desired inequality is

$$\left(\int_M \alpha^2 \beta^{d-2}\right)\left(\int_M \beta^2 \beta^{d-2}\right) \leq \left(\int_M \alpha\beta\beta^{d-2}\right)^2.$$

Since α and β are nef, this inequality holds by Lemma 4.36. □

As a result, the Heron-Rota-Welsh conjecture and the Mason's conjecture (i) (Conjecture 2.16 and 2.18 (i)) are proven.

Corollary 4.38. *For any loopless matroid M, the sequences $(|w_k(M)|)_k$ and $(f_k(M))_k$ are log-concave and unimodal.*

Example 4.39 ($U_{3,4}$, continued)**.** For the uniform matroid $M = U_{3,4}$ on $E = \{1, 2, 3, 4\}$, we have

$$\chi_M(q) = q^3 - 4q^2 + 6q - 3 \quad \text{and} \quad \overline{\chi}_M(q) = q^2 - 3q^2 + 3.$$

The coefficients of $\overline{\chi}_M(q)$ are exactly the degrees

$$\begin{aligned}
\int_M \alpha^2 &= \int_M (x_1 + x_{12} + x_{13} + x_{14})(x_2 + x_{12} + x_{23} + x_{24}) \\
&= \int_M x_1x_{12} + x_{12}(x_1 + x_{12}) \\
&= \int_M x_1x_{12} + x_{12}(x_3 + x_{23} + x_{24} - x_{14}) = \int_M x_1x_{12} = 1, \\
\int_M \alpha\beta &= \int_M (x_1 + x_{12} + x_{13} + x_{14})(x_2 + x_3 + x_4 + x_{23} + x_{24} + x_{34}) \\
&= \int_M x_2x_{12} + x_3x_{13} + x_4x_{14} = 3, \\
\int_M \beta^2 &= \int_M \beta(x_2 + x_3 + x_4 + x_{23} + x_{24} + x_{34}) \\
&= \int_M \sum_{2\notin F} x_F(x_2 + x_{23} + x_{24}) + \sum_{3\notin F} x_F(x_3 + x_{34}) + \sum_{4\notin F} x_Fx_4 \\
&= \int_M x_3x_{23} + x_4x_{24} + x_4x_{34} = 3
\end{aligned}$$

of $\alpha^2, \alpha\beta, \beta^2$, and are also equal to the cardinalities of $D_0(M) = \{()\}$, $D_1(M) = \{(2), (3), (4)\}$ and $D_2(M) = \{(3 \subsetneq 23), (4 \subsetneq 24), (4 \subsetneq 34)\}$.

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